

INDICES OF VECTOR FIELDS AND 1-FORMS ON SINGULAR VARIETIES

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ABSTRACT. We discuss different generalizations of the classical notion of the index of a singular point of a vector field to the case of vector fields or 1-forms on singular varieties, describe relations between them and formulae for their computation.

INTRODUCTION

An isolated singular point (zero) of a (continuous) vector field on a smooth manifold has a well known invariant — the index. A neighbourhood of a point P in a manifold M^n of dimension n can be identified with a neighbourhood of the origin in the affine (coordinate) space \mathbb{R}^n . A germ X of a vector field on $(\mathbb{R}^n, 0)$ can be written as $X(x) = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$. Let $B_\varepsilon^n(0)$ be the ball of radius ε centred at the origin in \mathbb{R}^n such that the vector field X is defined in a neighbourhood of this ball and has no zeros in it except at the origin. Let $S_\varepsilon^{n-1}(0)$ be the $(n-1)$ -dimensional sphere $\partial B_\varepsilon^n(0)$. The vector field X defines a map

$$\frac{X}{\|X\|} : S_\varepsilon^{n-1}(0) \rightarrow S_1^{n-1}.$$

The index $\text{ind}_P X$ of the vector field X at the point P is defined as the degree of the map $\frac{X}{\|X\|}$. One can see that it is independent of the chosen coordinates.

If the vector field X is smooth and its singular point P is non-degenerate, i.e. if $J_{X,P} := \det \left(\frac{\partial X_i}{\partial x_j}(0) \right) \neq 0$, then $\text{ind}_P X = \text{sgn } J_{X,P}$, i.e. it is equal to 1 if $J_{X,P} > 0$ and is equal to -1 if $J_{X,P} < 0$. The index of an arbitrary isolated singular point P of a smooth vector field X is equal to the number of non-degenerate singular points which split from the point P under a generic deformation \tilde{X} of the vector field X in a neighbourhood of the point P counted with the corresponding signs ($\text{sgn } J_{\tilde{X}, \tilde{P}}$). This follows, in particular, from the fact that the index of a vector field satisfies the law of conservation of number (see below).

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One of the most important properties of the index of a vector field is the Poincaré–Hopf theorem. Suppose that the manifold M is closed, i.e. compact without boundary, and that the vector field X has finitely many singular points on it.

Theorem 1 (Poincaré–Hopf). *The sum*

$$\sum_{P \in \text{Sing } X} \text{ind}_P X$$

of indices of singular points of the vector field X is equal to the Euler characteristic $\chi(M)$ of the manifold M .

If M is an n -dimensional complex analytic (compact) manifold, then its Euler characteristic $\chi(M)$ is equal to the characteristic number $\langle c_n(TM), [M] \rangle$, where $c_n(M)$ is the top Chern class of the manifold M . If a vector field X on a complex manifold M is holomorphic and a singular point P of it is non-degenerate, then the index $\text{ind}_P X$ is equal to $+1$. The index of an isolated singular point P of a holomorphic vector field X is positive. It is equal to the number of non-degenerate singular points which split from the point P under a generic deformation of the vector field X in a neighbourhood of the point P .

There exists an algebraic formula for the index $\text{ind}_P X$ of an isolated singular point of a holomorphic vector field. In local coordinates centred at the point P , let the vector field X be equal to $\sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$. Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the ring of germs of holomorphic functions of n variables.

Theorem 2.

$$\text{ind}_P X = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n,0} / (X_1, \dots, X_n),$$

where (X_1, \dots, X_n) is the ideal generated by the germs X_1, \dots, X_n .

This statement was known for a long time, however, it seems that the first complete proof appeared in [Pa67].

An algebraic formula for the index $\text{ind}_P X$ of an isolated (in fact for an algebraically isolated: see the definition below) singular point of a smooth (say, C^∞) vector field was given in [EL77, Kh77]. In local coordinates centred at the point P , let $X = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$. Let $\mathcal{E}_{\mathbb{R}^n,0}$ be the ring of germs of smooth (C^∞) functions of n variables.

Definition: The singular point P of the vector field v is algebraically isolated if the factor ring $\mathcal{E}_{\mathbb{R}^n,0} / (X_1, \dots, X_n)$ has a finite dimension as a vector space.

If a singular point of a vector field is algebraically isolated, then it is isolated. If, in the local coordinates x_1, \dots, x_n , the vector field X is (real) analytic (i.e., if the components X_i are analytic functions of x_1, \dots, x_n), one can consider its complexification $X_{\mathbb{C}}$ which is a holomorphic vector field in a neighbourhood of the origin in \mathbb{C}^n . In this case the singular point of

the vector field X is algebraically isolated if and only if the singular point of the vector field $X_{\mathbb{C}}$ (the origin) is isolated. Let

$$J_X = J_X(x) := \det \left(\frac{\partial X_i}{\partial x_j} \right)$$

be the *Jacobian* of the vector field X . One can prove that $J_X \neq 0$ in the local ring $R_X = \mathcal{E}_{\mathbb{R}^n,0}/(X_1, \dots, X_n)$. Moreover, the ring R_X has a one dimensional minimal ideal and this ideal is generated by the Jacobian J_X . Let $\ell : R_X \rightarrow \mathbb{R}$ be a linear function on the ring R_X (considered as a vector space) such that $\ell(J_X) > 0$. Consider the quadratic form Q on R_X defined by

$$Q(\varphi, \psi) = \ell(\varphi \cdot \psi).$$

Theorem 3 (Eisenbud-Levine-Khimshiashvili). *The index $\text{ind}_P X$ of the singular point P of the vector field X is equal to the signature $\text{sgn } Q$ of the quadratic form Q .*

For a proof of this theorem see also [AGV85].

In what follows we shall, in particular, discuss generalizations of the notion of the index of a vector field to real or complex analytic variety with singularities and problems of their computation.

1. 1-FORMS VERSUS VECTOR FIELDS

Traditionally the definition of the index and the corresponding results (say, the Poincaré–Hopf and the Eisenbud–Levine–Khimshiashvili theorems) are formulated for vector fields. However, instead of vector fields one can consider 1-forms. Using a Riemannian metric one can identify vector fields and 1-forms on a smooth (C^∞) manifold. Therefore on a smooth manifold all notions and statements concerning vector fields can be formulated for 1-forms as well. In particular, the index $\text{ind}_P \omega$ of a 1-form ω on a smooth n -dimensional manifold M at an isolated singular point (zero) P is defined (in local coordinates) as the degree of the map $\frac{\omega}{\|\omega\|} : S_\varepsilon^{n-1} \rightarrow S_1^{n-1}$, where S_1^{n-1} is the unit sphere in the dual space. In the local complex analytic situation one can identify germs of complex analytic vector fields and germs of complex analytic 1-forms using local coordinates: the vector field $\sum X_i \frac{\partial}{\partial z_i}$ corresponds to the 1-form $\sum X_i dz_i$. In the Poincaré–Hopf theorem for 1-forms on an n -dimensional complex manifold M the sum of indices of singular points of a 1-form is equal to $\langle c_n(T^*M), [M] \rangle = (-1)^n \chi(M)$. The only essential difference between vector fields and 1-forms in the smooth complex analytic case is that non-trivial complex analytic (global) vector fields and 1-forms exist, generally speaking, on different complex analytic manifolds: in the odd-dimensional case vector fields (respectively 1-forms) can exist only on manifolds with non-negative (respectively non-positive) Euler characteristic.

However, for the case of singular varieties the situation becomes quite different. In a series of papers instead of considering vector fields on a

variety we started to consider 1-forms. For 1-forms on germs of singular varieties (real or complex analytic) one can define several notions of indices usually corresponding to appropriate analogues for vector fields. However, the properties of the indices for vector fields and for 1-forms are different. For example, just as above, to an analytic vector field $X = \sum_{i=1}^N X_i \frac{\partial}{\partial x_i}$ on a real variety $(V, 0) \subset (\mathbb{R}^N, 0)$ (or on a complex analytic variety $(V, 0) \subset (\mathbb{C}^N, 0)$) one can associate the 1-form $\omega = \sum_{i=1}^N X_i dx_i$ (dependent on the choice of coordinates x_1, \dots, x_N in $(\mathbb{R}^N, 0)$ or in $(\mathbb{C}^N, 0)$). If the vector field X has an isolated singular point on $(V, 0)$, then, for a generic (!) choice of the coordinates x_1, \dots, x_N , the corresponding 1-form ω has an isolated singular point as well. This correspondence does not work in the other direction. Moreover, whereas for the radial index on a real analytic variety (see below) the index of a vector field coincides with that of the corresponding 1-form, this does not hold for other indices, say, for the so called GSV index (on an isolated complete intersection singularity: ICIS). In some cases one can say that the notion of an index of a 1-form (say, of a holomorphic one) is somewhat more natural than that of a vector field. For example, the notion of the GSV index of a 1-form (see below) is “more complex analytic” (does not use the complex conjugation for the definition) and “more geometric” (uses only objects of the same tensor type). Moreover the index of an isolated singular point of a holomorphic 1-form on a complex ICIS can be described as the dimension of an appropriate algebra. Finally, on a real ICIS the real index of a 1-form which is the differential of a germ of a function with an algebraically isolated singular point plus-minus the Euler characteristic of a (real) smoothing of the ICIS can be expressed in terms of the signature of a quadratic form on a space (certain space of thimbles), the dimension of which is equal to the (complex) index of the corresponding complexification.

The condition for a vector field to be tangent to a germ of a singular variety is a very restrictive one. For example, holomorphic vector fields with isolated zeros exist on a complex analytic variety with an isolated singularity [BG94], but not in general.

Example 1. Consider the surface X in \mathbb{C}^3 given by the equation

$$xy(x - y)(x + zy) = 0.$$

It has singularities on the line $\{x = y = 0\}$. It can be considered as a family of four lines in the (x, y) -plane with different cross ratios. Then any holomorphic vector field tangent to X vanishes on the line $\{x = y = 0\}$, because the translation along such a vector field has to preserve the cross ratio of the lines.

However, 1-forms with isolated singular points always exist.

The idea to consider indices of 1-forms instead of indices of vector fields (in some situations) was first formulated by V.I. Arnold ([Ar79], see also [Ar04]).

2. RADIAL INDEX

One can say that the notion of the radial index of a vector field or of a 1-form on a singular variety is a straightforward generalization of the usual index inspired by the Poincaré–Hopf theorem.

Let us start with the setting of (real: C^∞) manifolds with isolated singularities.

A *manifold with isolated singularities* is a topological space M which has the structure of a smooth (say, C^∞) manifold outside of a discrete set S (the *set of singular points* of M). A *diffeomorphism* between two such manifolds is a homeomorphism which sends the set of singular points onto the set of singular points and is a diffeomorphism outside of them. We say that M has a *cone-like singularity* at a (singular) point $P \in S$ if there exists a neighbourhood of the point P diffeomorphic to the cone over a smooth manifold W_P (W_P is called the *link* of the point P). In what follows we assume all manifolds to have only cone-like singularities. A (smooth or continuous) *vector field* or *1-form* on a manifold M with isolated singularities is a (smooth or continuous) vector field or 1-form respectively on the set $M \setminus S$ of regular points of M . The *set of singular points* S_X of a vector field X or the *set of singular points* S_ω of a 1-form ω on a (singular) manifold M is the union of the set of usual singular points of X or of ω respectively on $M \setminus S$ (i.e., points at which X or ω respectively tends to zero) and of the set S of singular points of M itself.

For an isolated *usual* singular point P of a vector field X or of a 1-form ω there is defined its index $\text{ind}_P X$ or $\text{ind}_P \omega$ respectively. If the manifold M is closed and has no singularities ($S = \emptyset$) and the vector field X or the 1-form ω on M has only isolated singularities, then the sum of these indices over all singular points is equal to the Euler characteristic $\chi(M)$ of the manifold M .

Let (M, P) be a cone-like singularity (i.e., a germ of a manifold with such a singular point). Let X be a vector field defined on an open neighbourhood U of the point P . Suppose that X has no singular points on $U \setminus \{P\}$. Let V be a closed cone-like neighbourhood of the point P in U ($V \cong CW_P$, $V \subset U$). On the cone $CW_P = (W_P \times I)/(W_P \times \{0\})$ ($I = [0, 1]$), there is defined a natural vector field $\partial/\partial t$ (t is the coordinate on I). Let X_{rad} be the corresponding vector field on V . Let \tilde{X} be a (continuous) vector field on U which coincides with X near the boundary ∂U of the neighbourhood U and with X_{rad} on V and has only isolated singular points.

Definition: The *radial index* $\text{ind}_{\text{rad}}(X; M, P)$ (or simply $\text{ind}_{\text{rad}} X$) of the vector field X at the point P is equal to

$$1 + \sum_{\tilde{P} \in S_{\tilde{X}} \setminus \{P\}} \text{ind}_{\tilde{P}} \tilde{X}$$

(the sum is over all singular points \tilde{P} of the vector field \tilde{X} except P itself).

Analogously, we define the radial index of a 1-form ω defined on an open neighbourhood U of the point P which has no singular points on $U \setminus \{P\}$. Let V again be a closed cone-like neighbourhood of P in U ($V \cong CW_P$, $V \subset U$). On the cone $CW_P = (W_P \times I)/(W_P \times \{0\})$ ($I = [0, 1]$) there is defined a natural 1-form dt . Let ω_{rad} be the corresponding 1-form on V . Let $\tilde{\omega}$ be a (continuous) 1-form on U which coincides with ω near the boundary ∂U of the neighbourhood U and with ω_{rad} on V and has only isolated singular points.

Definition: The *radial index* $\text{ind}_{\text{rad}}(\omega; M, P)$ (or simply $\text{ind}_{\text{rad}} \omega$) of the 1-form ω at the point P is equal to

$$1 + \sum_{\tilde{P} \in S_{\tilde{\omega}} \setminus \{P\}} \text{ind}_{\tilde{P}} \tilde{\omega}$$

(the sum is over all singular points \tilde{P} of the 1-form $\tilde{\omega}$ except P itself).

For a cone-like singularity at a point $P \in S$, the link W_P and thus the cone structure of a neighbourhood are, generally speaking, not well-defined (cones over different manifolds may be *locally* diffeomorphic). However it is not difficult to show that the indices $\text{ind}_{\text{rad}} X$ and $\text{ind}_{\text{rad}} \omega$ do not depend on the choice of a cone structure on a neighbourhood and on the choice of the vector field \tilde{X} and the 1-form $\tilde{\omega}$ respectively.

Example 2. The index of the radial vector field X_{rad} and of the radial 1-form ω_{rad} is equal to 1. The index of the vector field $(-X_{\text{rad}})$ and of the 1-form $(-\omega_{\text{rad}})$ is equal to $1 - \chi(W_P)$ where W_P is the link of the singular point P .

Proposition 1. *For a vector field X or a 1-form ω with isolated singular points on a closed manifold M with isolated singularities, the statement of Theorem 1 holds.*

Now let $(V, 0) \subset (\mathbb{R}^N, 0)$ be the germ of a real analytic variety, generally speaking, with a non-isolated singular point at the origin. Let $V = \bigcup_{i=1}^q V_i$ be a Whitney stratification of the germ $(V, 0)$. Let X be a continuous vector field on $(V, 0)$ (i.e., the restriction of a continuous vector field on $(\mathbb{R}^N, 0)$ tangent to V at each point) which has an isolated zero at the origin (on V). (Tangency to the stratified set V means that for each point $Q \in V$ the vector $X(Q)$ is tangent to the stratum V_i which contains the point Q .) Let V_i be a stratum of dimension k and let Q be a point of V_i . A neighbourhood of the point Q in V is diffeomorphic to the direct product of a linear space

\mathbb{R}^k and the cone CW_Q over a compact singular analytic variety W_Q . (Here a diffeomorphism between two stratified spaces is a homeomorphism which is a diffeomorphism on each stratum.) Let $\varepsilon > 0$ be small enough so that in the closed ball B_ε of radius ε centred at the origin in \mathbb{R}^N the vector field X has no zeros on $V \setminus \{0\}$. It is not difficult to show that there exists a (continuous) vector field \tilde{X} on $(V, 0)$ such that:

- (1) The vector field \tilde{X} is defined on $V \cap B_\varepsilon$.
- (2) \tilde{X} coincides with the vector field X in a neighbourhood of $V \cap S_\varepsilon$ in $V \cap B_\varepsilon$ (S_ε is the sphere ∂B_ε).
- (3) The vector field \tilde{X} has only a finite number of zeros.
- (4) Each point $Q \in U(0)$ with $\tilde{X}(Q) = 0$ has a neighbourhood diffeomorphic to $(\mathbb{R}^k, 0) \times CW_Q$ in which $\tilde{X}(y, z)$ ($y \in \mathbb{R}^k$, $z \in CW_Q$) is of the form $Y(y) + X_{\text{rad}}(z)$, where Y is a germ of a vector field on $(\mathbb{R}^k, 0)$ with an isolated singular point at the origin, X_{rad} is the radial vector field on the cone CW_Q .

Let $S_{\tilde{X}}$ be the set of zeros of the vector field \tilde{X} on $V \cap B_\varepsilon$. For a point $Q \in S_{\tilde{X}}$, let $\widetilde{\text{ind}}(Q) := \text{ind}(Y; \mathbb{R}^k, 0)$, where Y is the vector field on $(\mathbb{R}^k, 0)$ described above. If $k = 0$ (this happens at the origin if it is a stratum of the stratification $\{V_i\}$), we set $\widetilde{\text{ind}}(Q) := 1$.

Definition: The *radial index* $\text{ind}_{\text{rad}}(X; V, 0)$ of the vector field X on the variety V at the origin is the number

$$\sum_{Q \in S_{\tilde{X}}} \widetilde{\text{ind}}(Q).$$

One can say that the idea of this definition goes back to M.-H. Schwartz who defined this index for so called radial vector fields [Schw65, Schw86a, Schw86b]; cf. also [KT99]. It was used to define characteristic classes for singular varieties, see e.g. the surveys [Br00, Se02].

Let ω be an (arbitrary continuous) 1-form on a neighbourhood of the origin in \mathbb{R}^N with an isolated singular point on $(V, 0)$ at the origin. (A point $Q \in V$ is a singular point of the 1-form ω on V if it is singular for the restriction of the 1-form ω to the stratum V_i which contains the point Q ; if the stratum V_i is zero-dimensional, the point Q is always singular.) Let $\varepsilon > 0$ be small enough so that in the closed ball B_ε of radius ε centred at the origin in \mathbb{R}^N the 1-form ω has no singular points on $V \setminus \{0\}$. We say that a 1-form ω on a germ $(W, 0)$ is radial if for any analytic curve $\gamma : (\mathbb{R}, 0) \rightarrow (W, 0)$ different from the trivial one ($\gamma(t) \not\equiv 0$) the value $\omega(\dot{\gamma}(t))$ of the 1-form ω on the tangent vector to the curve γ is positive for positive t small enough. It is easy to see that there exists a 1-form $\tilde{\omega}$ on \mathbb{R}^N such that:

- (1) The 1-form $\tilde{\omega}$ coincides with the 1-form ω on a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$.

- (2) the vector field \tilde{X} has only a finite number of zeros;
- (3) In a neighbourhood of each singular point $Q \in (V \cap B_\varepsilon) \setminus \{0\}$, $Q \in V_i$, $\dim V_i = k$, the 1-form $\tilde{\omega}$ looks as follows. There exists a (local) analytic diffeomorphism $h : (\mathbb{R}^N, \mathbb{R}^k, 0) \rightarrow (\mathbb{R}^N, V_i, Q)$ such that $h^*\tilde{\omega} = \pi_1^*\tilde{\omega}_1 + \pi_2^*\tilde{\omega}_2$, where π_1 and π_2 are the natural projections $\pi_1 : \mathbb{R}^N \rightarrow \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^N \rightarrow \mathbb{R}^{N-k}$ respectively, $\tilde{\omega}_1$ is the germ of a 1-form on $(\mathbb{R}^k, 0)$ with an isolated singular point at the origin, and $\tilde{\omega}_2$ is a radial 1-form on $(\mathbb{R}^{N-k}, 0)$.

Remark. One can demand that the 1-form $\tilde{\omega}_1$ has a non-degenerate singular point (and therefore $\text{ind}(\tilde{\omega}_1, \mathbb{R}^k, 0) = \pm 1$), however, this is not necessary for the definition.

Let $S_{\tilde{\omega}}$ be the set of singular points of the 1-form $\tilde{\omega}$ on $V \cap B_\varepsilon$. For a point $Q \in S_{\tilde{\omega}}$, let $\widetilde{\text{ind}}(Q) := \text{ind}(\tilde{\omega}_1; \mathbb{R}^k, 0)$. If $k = 0$, we set $\widetilde{\text{ind}}(Q) := 1$.

Definition: The *radial index* $\text{ind}_{\text{rad}}(\omega; V, 0)$ of the 1-form ω on the variety V at the origin is the sum

$$\sum_{Q \in S_{\tilde{\omega}}} \widetilde{\text{ind}}(Q).$$

One can show that these notions are well defined (for the radial index of a 1-form see [EG05a]).

Just because of the definition, the (radial) index satisfies the law of conservation of number. For a 1-form this means the following: if a 1-form ω' with isolated singular points on V is close to the 1-form ω , then

$$\text{ind}_{\text{rad}}(\omega; V, 0) = \sum_{Q \in \text{Sing } \omega'} \text{ind}_{\text{rad}}(\omega'; V, 0)$$

where the sum on the right hand side is over all singular points Q of the 1-form ω' on V in a neighbourhood of the origin.

The radial index generalizes the usual index for vector fields or 1-forms with isolated singularities on a smooth manifold. In particular one has a generalization of the Poincaré-Hopf theorem:

Theorem 4 (Poincaré-Hopf). *For a compact real analytic variety V and a vector field X or a 1-form ω with isolated singular points on V , one has*

$$\sum_Q \text{ind}_{\text{rad}}(X; V, Q) = \sum_Q \text{ind}_{\text{rad}}(\omega; V, Q) = \chi(V)$$

where $\chi(V)$ denotes the Euler characteristic of the set (variety) V .

In the case of radial vector fields, this theorem is due to M.-H. Schwartz [Schw65, Schw86a, Schw86b, Schw91] (see also [BS81, ASV98, KT99]). For a proof of this theorem for the case of 1-forms see [EG05a].

Now let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a complex analytic variety of pure dimension n . Let ω be a (complex and, generally speaking, continuous) 1-form on a neighbourhood of the origin in \mathbb{C}^N . In fact there is a one-to-one correspondence between complex 1-forms on a complex manifold M^n (say, on

\mathbb{C}^N) and real 1-forms on it (considered as a real $2n$ -dimensional manifold). Namely, to a complex 1-form ω one associates the real 1-form $\eta = \operatorname{Re} \omega$; the 1-form ω can be restored from η by the formula $\omega(v) = \eta(v) - i\eta(iv)$ for $v \in T_x M^n$. This means that the index of the real 1-form $\operatorname{Re} \omega$ is an invariant of the complex 1-form ω itself. However, on a smooth manifold $\operatorname{ind}_{M^n, x} \operatorname{Re} \omega$ does not coincide with the usual index of the singular point x of the 1-form ω , but differs from it by the coefficient $(-1)^n$. (E.g., the index of the (complex analytic) 1-form $\omega = \sum_{j=1}^n x_j dx_j$ ((x_1, \dots, x_n) being the coordinates of \mathbb{C}^n) is equal to 1, whence the index of the real 1-form $\operatorname{Re} \omega = \sum_{j=1}^n u_j du_j - \sum_{j=1}^n v_j dv_j$ ($x_j = u_j + iv_j$) is equal to $(-1)^n$.) This explains the following definition.

Definition: The (*complex radial*) index $\operatorname{ind}_{\operatorname{rad}}^{\mathbb{C}}(\omega; V, 0)$ of the complex 1-form ω on an n -dimensional variety V at the origin is $(-1)^n$ times the index of the real 1-form $\operatorname{Re} \omega$ on V :

$$\operatorname{ind}_{\operatorname{rad}}^{\mathbb{C}}(\omega; V, 0) = (-1)^n \operatorname{ind}_{\operatorname{rad}}(\operatorname{Re} \omega; V, 0).$$

3. GSV INDEX

This generalization of the index makes sense for varieties which have isolated complete intersection singularities (in particular, for hypersurfaces with isolated singularities).

Let $V \subset (\mathbb{C}^{n+k}, 0)$ be an n -dimensional isolated complete intersection singularity (ICIS) defined by equations $f_1 = \dots = f_k = 0$ (i.e. $V = f^{-1}(0)$), where $f = (f_1, \dots, f_k) : (\mathbb{C}^{n+k}, 0) \rightarrow (\mathbb{C}^k, 0)$, $\dim V = n$ and let $X = \sum_{i=1}^{n+k} X_i \frac{\partial}{\partial z_i}$ be a germ of a (continuous) vector field on $(\mathbb{C}^{n+k}, 0)$ tangent to the ICIS V . (The latter means that $X(z) \in T_z V$ for all $z \in V \setminus \{0\}$.) Suppose that the vector field X does not vanish on V in a punctured neighbourhood of the origin. In this situation the following index (called the GSV index after X. Gómez-Mont, J. Seade, and A. Verjovsky) is defined.

Let $B_\varepsilon \subset \mathbb{C}^{n+k}$ be the ball of radius ε centred at the origin with (positive) ε small enough so that all the functions f_i ($i = 1, \dots, k$) and the vector field X are defined in a neighbourhood of B_ε , V is transversal to the sphere $S_\varepsilon = \partial B_\varepsilon$, and the vector field X has no zeros on V inside the ball B_ε except (possibly) at the origin. Let $K = V \cap S_\varepsilon$ be the link of the ICIS $(V, 0)$. The manifold K is $(2n-1)$ -dimensional and has a natural orientation as the boundary of the complex manifold $V \cap B_\varepsilon \setminus \{0\}$.

Let $\mathcal{M}(p, q)$, $p \geq q$, be the space of $p \times q$ matrices with complex entries and let $D_{p,q}$ be the subspace of $\mathcal{M}(p, q)$ consisting of matrices of rank less than q . The subset $D_{p,q}$ is an irreducible subvariety of $\mathcal{M}(p, q)$ of codimension $p - q + 1$. The complement $W_{p,q} = \mathcal{M}(p, q) \setminus D_{p,q}$ is the Stiefel manifold of q -frames (collections of q linearly independent vectors) in \mathbb{C}^p . It is known that the Stiefel manifold $W_{p,q}$ is $2(p-q)$ -connected and $H_{2(p-q)+1}(W_{p,q}) \cong \mathbb{Z}$ (see,

e.g., [Hu75]). The latter fact also proves that $D_{p,q}$ is irreducible. Since $W_{p,q}$ is the complement of an irreducible complex analytic subvariety of codimension $p - q + 1$ in $\mathcal{M}(p, q) \cong \mathbb{C}^{pq}$, there is a natural choice of a generator of the homology group $H_{2(p-q)+1}(W_{p,q}) \cong \mathbb{Z}$. Namely, the ("positive") generator is the boundary of a small ball in a smooth complex analytic slice transversal to $D_{p,q}$ at a non-singular point. Therefore a map from an oriented smooth (C^∞) closed $(2(p - q) + 1)$ -dimensional manifold to $W_{p,q}$ has a degree.

Define the *gradient vector field* $\text{grad } f_i$ of a function germ f_i by

$$\text{grad } f_i = \left(\frac{\partial f_i}{\partial z_1}, \dots, \frac{\partial f_i}{\partial z_{n+k}} \right)$$

($\text{grad } f_i$ depends on the choice of the coordinates z_1, \dots, z_{n+k}). One has a map

$$(1) \quad \Psi = (X, \text{grad } f_1, \dots, \text{grad } f_k) : K \rightarrow W_{n+k, k+1}$$

from the link K to the Stiefel manifold $W_{n+k, k+1}$.

Definition: The *GSV index* $\text{ind}_{\text{GSV}}(X; V, 0)$ of the vector field X on the ICIS V at the origin is the degree of the map

$$\Psi : K \rightarrow W_{n+k, k+1}.$$

Remark. Note that one uses the complex conjugation for this definition and the components of the discussed map are of different tensor nature. Whereas X is a vector field, $\text{grad } f_i$ is more similar to a covector.

This index was first defined in [GSV91] for vector fields on isolated hypersurface singularities. In [SS96] it was generalized to vector fields on ICIS.

It is convenient to consider the map Ψ as a map from V to $\mathcal{M}(n+k, k+1)$ defined by the formula (1) (in a neighbourhood of the ball B_ε). It maps the complement of the origin in V to the Stiefel manifold $W_{n+k, k+1}$. This description leads to the following definition of the GSV index.

Proposition 2. *The GSV index $\text{ind}_{\text{GSV}}(X; V, 0)$ of the vector field X on the ICIS V at the origin is equal to the intersection number $(\Psi(V) \circ D_{n+k, k+1})$ of the image $\Psi(V)$ of the ICIS V under the map Ψ and the variety $D_{n+k, k+1}$ at the origin.*

Note that, even if the vector field X is holomorphic, $\Psi(V)$ is not, generally speaking, a complex analytic variety.

Another definition/description of the GSV index $\text{ind}_{\text{GSV}}(X; V, 0)$ can be given in the following way. Let $\tilde{V} = V_t = f^{-1}(t) \cap B_\varepsilon$, where $t \in \mathbb{C}^k$, $0 < \|t\| \ll \varepsilon$, be the Milnor fibre of the ICIS V , i.e. a smoothing of it. For t small enough in a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ the manifolds $V = V_0$ and V_t "almost coincide". This gives an up to isotopy well defined vector field \tilde{X} on the manifold \tilde{V} in a neighbourhood of its boundary $f^{-1}(t) \cap S_\varepsilon$ (generally speaking, not a complex analytic one). Let us extend this vector

field to a vector field (also denoted by \tilde{X}) on the entire manifold \tilde{V} with only isolated zeros.

Proposition 3. *One has*

$$\text{ind}_{\text{GSV}}(X; V, 0) = \sum_{Q \in \text{Sing } \tilde{X}} \text{ind}_Q \tilde{X}.$$

This definition can be easily generalized to a germ of a complex analytic variety with an isolated singularity and with a fixed smoothing. For example, J. Seade defined in this way an index for a singular point of a vector field on a complex analytic surface with a normal smoothable Gorenstein singularity [Se87]. For a singularity which is not an ICIS, however, it is possible that such a smoothing does not exist or there may be a number of different smoothings what leads to the situation that the index is not well defined. These difficulties cannot be met for curve singularities. Thus in this situation the corresponding index is well defined (cf. [Gor00]). For a general variety with an isolated singularity, this ambiguity can be avoided for germs of functions (i.e. for their differentials in our terms) by defining the corresponding index as a certain residue, what is done in [IS03].

Now let V be a compact (say, projective) variety all singular points of which are local ICIS and let X be a vector field on V with isolated singular points. One has the following statement.

Proposition 4. *One has*

$$\sum_{Q \in \text{Sing } \omega} \text{ind}_{\text{GSV}}(\omega; V, Q) = \chi(\tilde{V}),$$

where \tilde{V} is a smoothing of the variety (local complete intersection) V .

A similar construction can be considered in the real setting. Namely, let $V = f^{-1}(0) \subset (\mathbb{R}^{n+k}, 0)$ be a germ of a real n -dimensional ICIS, $f = (f_1, \dots, f_k) : (\mathbb{R}^{n+k}, 0) \rightarrow (\mathbb{R}^k, 0)$ is a real analytic map, and let $X = \sum_{i=1}^{n+k} X_i dz_i$ be a germ of a (continuous) vector field on $(\mathbb{R}^{n+k}, 0)$ tangent to the variety V outside of the origin. Just as above one defines a map Ψ from the link $K = V \cap S_\varepsilon^{n+k-1}$ of the ICIS $(V, 0)$ to the Stiefel manifold $W_{n+k, k+1}^{\mathbb{R}}$ of $(k+1)$ -frames in \mathbb{R}^{n+k} . The Stiefel manifold $W_{n+k, k+1}^{\mathbb{R}}$ is $(n-2)$ -connected and its first non-trivial homology group $H_{n-1}(W_{n+k, k+1}^{\mathbb{R}}; \mathbb{Z})$ is isomorphic to the group \mathbb{Z} of integers for n odd and to the group \mathbb{Z}_2 of order 2 for n even. Therefore the map Ψ has a degree defined as an integer or as an integer modulo 2 depending on the parity of the dimension n . If the manifold $V \setminus \{0\}$ is not connected, the construction can be applied to each connected component of it giving a set of degrees (a multi-degree). This invariant (called *real GSV index* of the vector field X on the real ICIS V) was introduced and studied in [ASV98].

In [EG01, EG03a] the notion of the GSV index was adapted to the case of a 1-form. Let $\omega = \sum A_i dx_i$ ($A_i = A_i(x)$) be a germ of a continuous 1-form on $(\mathbb{C}^{n+k}, 0)$ which as a 1-form on the ICIS V has (at most) an isolated singular point at the origin (thus it does not vanish on the tangent space $T_P V$ to the variety V at all points P from a punctured neighbourhood of the origin in V). The 1-forms $\omega, df_1, \dots, df_k$ are linearly independent for all $P \in K$. Thus one has a map

$$\Psi = (\omega, df_1, \dots, df_k) : K \rightarrow W_{n+k, k+1}.$$

Here $W_{n+k, k+1}$ is the Stiefel manifold of $(k+1)$ -frames in the space dual to \mathbb{C}^{n+k} .

Definition: The *GSV index* $\text{ind}_{\text{GSV}}(\omega; V, 0)$ of the 1-form ω on the ICIS V at the origin is the degree of the map

$$\Psi : K \rightarrow W_{n+k, k+1}.$$

Just as above Ψ can be considered as a map from the ICIS V to the space $\mathcal{M}(n+k, k+1)$ of $(n+k) \times (k+1)$ matrices. If the 1-form ω is holomorphic, the map Ψ and the set $\Psi(V)$ are complex analytic.

The obvious analogues of Propositions 2, 3, and 4 hold.

There exists an algebraic formula for the index $\text{ind}_{\text{GSV}}(\omega; V, 0)$ of a holomorphic 1-form ω on an ICIS V which gives it as the dimension of a certain algebra (see Section 6).

In [BSS05a], there was defined a generalization of the notion of the GSV index for a 1-form on a complete intersection singularity $V = f^{-1}(0) \subset (\mathbb{C}^{n+k}, 0)$ if the map $f = (f_1, \dots, f_k) : (\mathbb{C}^{n+k}, 0) \rightarrow (\mathbb{C}^k, 0)$, defining the singularity satisfies Thom's a_f regularity condition. (This condition implies that the Milnor fibre of the complete intersection singularity V is well-defined.)

4. HOMOLOGICAL INDEX

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be any germ of an analytic variety of pure dimension n with an isolated singular point at the origin. Suppose X is a complex analytic vector field tangent to $(V, 0)$ with an isolated singular point at the origin.

Let $\Omega_{V,0}^k$ be the module of germs of differentiable k -forms on $(V, 0)$, i.e. the factor of the module $\Omega_{\mathbb{C}^N,0}^k$ of k -forms on $(\mathbb{C}^N, 0)$ by the submodule generated by $f \cdot \Omega_{\mathbb{C}^N,0}^k$ and $df \wedge \Omega_{\mathbb{C}^N,0}^k$ for all f from the ideal of functions vanishing on $(V, 0)$. Consider the Koszul complex $(\Omega_{V,0}^\bullet, X)$:

$$0 \longleftarrow \mathcal{O}_{V,0} \xleftarrow{X} \Omega_{V,0}^1 \xleftarrow{X} \dots \xleftarrow{X} \Omega_{V,0}^n \longleftarrow 0,$$

where the arrows are given by contraction with the vector field X . The sheaves $\Omega_{V,0}^i$ are coherent sheaves and the homology groups of the complex

$(\Omega_{V,0}^\bullet, X)$ are concentrated at the origin and therefore are finite dimensional. The following definition is due to X. Gómez-Mont [Góm98].

Definition: The *homological index* $\text{ind}_{\text{hom}}(X; V, 0) = \text{ind}_{\text{hom}} X$ of the vector field X on $(V, 0)$ is the Euler characteristic of the above complex:

$$(2) \quad \text{ind}_{\text{hom}}(X; V, 0) = \text{ind}_{\text{hom}} X = \sum_{j=0}^n (-1)^j h_j(\Omega_{V,0}^\bullet, X),$$

where $h_j(\Omega_{V,0}^\bullet, X)$ is the dimension of the corresponding homology group as a vector space over \mathbb{C} .

The homological index satisfies the law of conservation of number [Góm98, Theorem 1.2]. In the case when V is a hypersurface, Gómez-Mont has shown that the homological index is equal to the GSV index.

Given a holomorphic 1-form ω on $(V, 0)$ with an isolated singularity, we consider the complex $(\Omega_{V,0}^\bullet, \wedge\omega)$:

$$0 \longrightarrow \mathcal{O}_{V,0} \xrightarrow{\wedge\omega} \Omega_{V,0}^1 \xrightarrow{\wedge\omega} \dots \xrightarrow{\wedge\omega} \Omega_{V,0}^n \longrightarrow 0,$$

where the arrows are given by the exterior product by the form ω .

This complex is the dual of the Koszul complex considered above. It was used by G.M. Greuel in [Gr75] for complete intersections. In [EGS04] the definition of Gómez-Mont was adapted to this case.

Definition: The *homological index* $\text{ind}_{\text{hom}}(\omega; V, 0) = \text{ind}_{\text{hom}} \omega$ of the 1-form ω on $(V, 0)$ is $(-1)^n$ times the Euler characteristic of the above complex:

$$(3) \quad \text{ind}_{\text{hom}}(\omega; V, 0) = \text{ind}_{\text{hom}} \omega = \sum_{j=0}^n (-1)^{n-j} h_j(\Omega_{V,0}^\bullet, \wedge\omega),$$

where $h_j(\Omega_{V,0}^\bullet, \wedge\omega)$ is the dimension of the corresponding homology group as a vector space over \mathbb{C} .

In [EGS04], there was proved the following statement.

Theorem 5. *Let ω be a holomorphic 1-form on V with an isolated singularity at the origin 0.*

- (i) *If V is smooth, then $\text{ind}_{\text{hom}} \omega$ equals the usual index of the holomorphic 1-form ω .*
- (ii) *The homological index satisfies the law of conservation of number: if ω' is a holomorphic 1-form on V close to ω , then:*

$$\text{ind}_{\text{hom}}(\omega; V, 0) = \text{ind}_{\text{hom}}(\omega'; V, 0) + \sum \text{ind}_{\text{hom}}(\omega'; V, x),$$

where the sum on the right hand side is over all those points x in a small punctured neighbourhood of the origin 0 in V where the form ω' vanishes.

- (iii) *If $(V, 0)$ is an isolated complete intersection singularity, then the homological index $\text{ind}_{\text{hom}} \omega$ coincides with the GSV index $\text{ind}_{\text{GSV}} \omega$.*

Statement (ii) follows from [GG02].

Remark. We notice that one has an invariant for functions on $(V, 0)$ with an isolated singularity at the origin defined by $f \mapsto \text{ind}_{\text{hom}} df$. By the theorem above, if $(V, 0)$ is an isolated complete intersection singularity, this invariant counts the number of critical points of the function f on a Milnor fibre of the ICIS V .

Remark. Let $(C, 0)$ be a curve singularity and let $(\bar{C}, \bar{0})$ be its normalization. Let $\tau = \dim \text{Ker}(\Omega_{C,0}^1 \rightarrow \Omega_{\bar{C},\bar{0}}^1)$, $\lambda = \dim \omega_{C,0}/c(\Omega_{C,0}^1)$, where $\omega_{C,0}$ is the dualizing module of Grothendieck, $c : \Omega_{C,0}^1 \rightarrow \omega_{C,0}$ is the class map (see [BG80]). In [MvS01] there is considered a Milnor number of a function f on a curve singularity introduced by V. Goryunov [Gor00]. One can see that this Milnor number can be defined for a 1-form ω with an isolated singularity on $(C, 0)$ as well (as $\dim \omega_{C,0}/\omega \wedge \mathcal{O}_{C,0}$) and is equal to $\text{ind}_{\text{hom}} \omega + \lambda - \tau$.

The laws of conservation of numbers for the homological and the radial indices of 1-forms together with the fact that these two indices coincide on smooth varieties imply that their difference is a locally constant, and therefore constant, function on the space of 1-forms on V with isolated singularities at the origin. Therefore one has the following statement.

Proposition 5. *Let $(V, 0)$ be a germ of a complex analytic space of pure dimension n with an isolated singular point at the origin. Then the difference*

$$\text{ind}_{\text{hom}} \omega - \text{ind}_{\text{rad}} \omega$$

between the homological and the radial index does not depend on the 1-form ω .

If $(V, 0)$ is an ICIS, this difference is equal to the Milnor number of $(V, 0)$. Together with this fact proposition 5 permits to consider the difference

$$\nu(V, 0) := \text{ind}_{\text{hom}}(\omega; V, 0) - \text{ind}_{\text{rad}}(\omega; V, 0)$$

as a generalized Milnor number of the singularity $(V, 0)$.

There are other invariants of isolated singularities of complex analytic varieties which coincide with the Milnor number for isolated complete intersection singularities. One of them is $(-1)^n$ times the reduced Euler characteristic (i.e., the Euler characteristic minus 1) of the absolute de Rham complex of $(V, 0)$:

$$0 \longrightarrow \mathcal{O}_{V,0} \xrightarrow{d} \Omega_{V,0}^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_{V,0}^n \longrightarrow 0 :$$

$$\bar{\chi}(V, 0) := \sum_{i=0}^n (-1)^{n-i} h_i(\Omega_{V,0}^\bullet, d) - (-1)^n.$$

In [EGS04] it was shown that:

Theorem 6. *One has*

$$\nu(V, 0) = \bar{\chi}(V, 0)$$

if

- (i) $(V, 0)$ *is a curve singularity,*
- (ii) $(V, 0) \subset (\mathbb{C}^{d+1}, 0)$ *is the cone over the rational normal curve in \mathbb{CP}^d .*

Statement (i) implies that the invariant $\nu(V, 0)$ is different from the Milnor number introduced by R.-O. Buchweitz and G.-M. Greuel [BG80] for curve singularities. Statement (ii) was obtained with the help of H.-Ch. Graf von Bothmer and R.-O. Buchweitz. For $d = 4$ this is Pinkham's example [Pi74] of a singularity which has smoothings with different Euler characteristics.

5. EULER OBSTRUCTION

The idea of the Euler obstruction emerged from [Mac74] where the Euler obstruction of a singular point of a complex analytic variety was defined (the definition was formulated in terms of obstruction theory in [BS81]).

The Euler obstruction of an isolated singular point (zero) of a vector field on a (singular) complex analytic variety was essentially defined in [BMPS04] (though formally speaking there it was defined only for holomorphic functions, i.e. for corresponding gradient vector fields). The main result of [BMPS04] (see Theorem 7 below) gives a relation between the Euler obstruction of a function germ on a singular variety V and the Euler obstructions of singular points of V itself.

In [BMPS04], there is introduced the notion of the local Euler obstruction of a holomorphic function with an isolated critical point on the germ of a complex analytic variety. It is defined as follows.

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a purely n -dimensional complex analytic variety with a Whitney stratification $V = \bigcup_{i=1}^q V_i$ and let f be a holomorphic function defined in a neighbourhood of the origin in \mathbb{C}^N with an isolated singular point on V at the origin. Let $\varepsilon > 0$ be small enough such that the function f has no singular points on $V \setminus \{0\}$ inside the ball B_ε . Let $\text{grad } f$ be the gradient vector field of f as defined in Section 3. Since f has no singular points on $V \setminus \{0\}$ inside the ball B_ε , the angle of $\text{grad } f(x)$ and the tangent space $T_x V_i$ to a point $x \in V_i \setminus \{0\}$ is less than $\pi/2$. Denote by $\zeta_i(x) \neq 0$ the projection of $\text{grad } f(x)$ to the tangent space $T_x V_i$.

Following the construction in [BMPS04], the vector fields ζ_i can be glued together to obtain a stratified vector field $\text{grad}_V f$ on V such that $\text{grad}_V f$ is homotopic to the restriction of $\text{grad } f$ to V and satisfies $\text{grad}_V f(x) \neq 0$ unless $x = 0$.

Let $\nu : \widehat{V} \rightarrow V$ be the Nash transformation of the variety V defined as follows. Let $G(n, N)$ be the Grassmann manifold of n -dimensional vector subspaces of \mathbb{C}^N . For a suitable neighbourhood U of the origin in \mathbb{C}^N , there is a natural map $\sigma : V_{\text{reg}} \cap U \rightarrow U \times G(n, N)$ which sends a point x to $(x, T_x V_{\text{reg}})$ (V_{reg} is the non-singular part of V). The Nash transform \widehat{V} is

the closure of the image $\text{Im } \sigma$ of the map σ in $U \times G(n, N)$. The Nash bundle \widehat{T} over \widehat{V} is a vector bundle of rank n which is the pullback of the tautological bundle on the Grassmann manifold $G(n, N)$. There is a natural lifting of the Nash transformation to a bundle map from the Nash bundle \widehat{T} to the restriction of the tangent bundle $T\mathbb{C}^N$ of \mathbb{C}^N to V . This is an isomorphism of \widehat{T} and $TV_{\text{reg}} \subset T\mathbb{C}^N$ over the regular part V_{reg} of V .

The vector field $\text{grad}_V f$ gives rise to a section $\widehat{\zeta}$ of the Nash bundle \widehat{T} over the Nash transform \widehat{V} without zeros outside of the preimage of the origin.

Definition: [BMPS04] The *local Euler obstruction* $\text{Eu}_{V,0} f$ of the function f on V at the origin is the obstruction to extend the non-zero section $\widehat{\zeta}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to the preimage of its interior, more precisely its value (as an element of $H^{2n}(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))$) on the fundamental class of the pair $(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))$.

The word *local* will usually be omitted.

Remark. The local Euler obstruction can also be defined for the restriction to V of a real analytic function on \mathbb{R}^{2N} . In particular, one can take for f the squared distance on V to the origin. In this case, the invariant $\text{Eu}_{V,0} f$ is the usual local Euler obstruction $\text{Eu}_V(0)$ of the variety $(V, 0)$ defined in [Mac74, BS81, BLS00].

Denote by $M_f = M_{f,t_0}$ the Milnor fibre of f , i.e. the intersection $V \cap B_\varepsilon \cap f^{-1}(t_0)$ for a regular value t_0 of f close to 0. In [BMPS04, Theorem 3.1] the following result is proved.

Theorem 7. *Let $f : (V, 0) \rightarrow (\mathbb{C}, 0)$ have an isolated singularity at $0 \in V$. Then*

$$\text{Eu}_V(0) = \left(\sum_{i=1}^q \chi(M_f \cap V_i) \cdot \text{Eu}_V(V_i) \right) + \text{Eu}_{V,0} f,$$

where $\text{Eu}_V(V_i)$ is the value of the Euler obstruction of V at any point of V_i , $i = 1, \dots, q$.

One can also define the local Euler obstruction of a stratified vector field on a germ of a complex analytic variety $(V, 0)$ with a Whitney stratification [BSS05b]. Let V_i be a stratum of the Whitney stratification with $0 \in V_i$ and X_i be a vector field on V_i with an isolated singularity at 0. The Proportionality Theorem of [BS81, BSS05b] states that the local Euler obstruction of a radial extension X of X_i at 0 is equal to the local Euler obstruction $\text{Eu}_V(0)$ of $(V, 0)$ times the radial index $\text{ind}_{\text{rad}}(X; V, 0)$ of X .

In [EG05a] the definition of the local Euler obstruction of a function was adapted to the case of a 1-form. Let ω be a 1-form on a neighbourhood of the origin in \mathbb{C}^N with an isolated singular point on V at the origin. Let $\varepsilon > 0$ be small enough such that the 1-form ω has no singular points on $V \setminus \{0\}$ inside the ball B_ε .

The 1-form ω gives rise to a section $\widehat{\omega}$ of the dual Nash bundle \widehat{T}^* over the Nash transform \widehat{V} without zeros outside of the preimage of the origin.

Definition: The *local Euler obstruction* $\text{Eu}_{V,0}\omega$ of the 1-form ω on V at the origin is the obstruction to extend the non-zero section $\hat{\omega}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to the preimage of its interior, more precisely its value (as an element of the cohomology group $H^{2n}(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))$) on the fundamental class of the pair $(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))$.

Remark. The local Euler obstruction can also be defined for a real 1-form on the germ of a real analytic variety if the last one is orientable in an appropriate sense.

Example 3. Let $\omega = df$ for the germ f of a holomorphic function on $(\mathbb{C}^N, 0)$. Then $\text{Eu}_{V,0}df$ differs from the Euler obstruction $\text{Eu}_{f,V}(0)$ of the function f by the sign $(-1)^n$. The reason is that for the germ of a holomorphic function with an isolated critical point on $(\mathbb{C}^n, 0)$ one has $\text{Eu}_{f,V}(0) = (-1)^n \mu_f$ (see [BMPS04, Remark 3.4]), whence $\text{Eu}_{V,0}df = \mu_f$ (μ_f is the Milnor number of the germ f). E.g., for $f(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2$ the obstruction $\text{Eu}_{f,V}(0)$ is the index of the vector field $\sum_{i=1}^n x_i \partial / \partial x_i$ (which is equal to $(-1)^n$), but the obstruction $\text{Eu}_{V,0}df$ is the index of the (holomorphic) 1-form $\sum_{i=1}^n x_i dx_i$ which is equal to 1.

The Euler obstruction of a 1-form satisfies the law of conservation of number (just as the radial index). Moreover, on a smooth variety the Euler obstruction and the radial index coincide. This implies the following statement (cf. Theorem 7). We set $\overline{\chi}(Z) := \chi(Z) - 1$ and call it the *reduced* (modulo a point) Euler characteristic of the topological space Z (though, strictly speaking, this name is only correct for a non-empty space Z).

Proposition 6. *Let $(V, 0) \subset (\mathbb{C}^N, 0)$ have an isolated singular point at the origin and let $\ell : \mathbb{C}^N \rightarrow \mathbb{C}$ be a generic linear function. Then*

$$\text{ind}_{\text{rad}}(\omega; V, 0) - \text{Eu}_{V,0}\omega = \text{ind}_{\text{rad}}(d\ell; V, 0) = (-1)^{n-1} \overline{\chi}(M_\ell),$$

where M_ℓ is the Milnor fibre of the linear function ℓ on V . In particular

$$\text{Eu}_{V,0}df = (-1)^n (\chi(M_\ell) - \chi(M_f)).$$

An analogue of the Proportionality Theorem for 1-forms was proved in [BSS05a].

Now let $(V, 0) \subset (\mathbb{C}^N, 0)$ be an arbitrary germ of an analytic variety with a Whitney stratification $V = \bigcup_{i=0}^q V_i$, where we suppose that $V_0 = \{0\}$. For a stratum V_i , $i = 0, \dots, q$, let N_i be the normal slice in the variety V to the stratum V_i ($\dim N_i = \dim V - \dim V_i$) at a point of the stratum V_i and let n_i be the index of a generic (non-vanishing) 1-form $d\ell$ on N_i :

$$n_i = (-1)^{\dim N_i - 1} \overline{\chi}(M_{\ell|_{N_i}}).$$

In particular for an open stratum V_i of V , N_i is a point and $n_i = 1$. The strata V_i of V are partially ordered: $V_i \prec V_j$ (we shall write $i \prec j$) iff $V_i \subset \overline{V_j}$ and $V_i \neq V_j$; $i \preceq j$ iff $i \prec j$ or $i = j$. For two strata V_i and V_j

with $i \preceq j$, let N_{ij} be the normal slice of the variety $\overline{V_j}$ to the stratum V_i at a point of it ($\dim N_{ij} = \dim V_j - \dim V_i$, N_{ii} is a point) and let n_{ij} be the index of a generic 1-form $d\ell$ on N_{ij} : $n_{ij} = (-1)^{\dim N_{ij}-1} \overline{\chi}(M_{\ell|N_{ij}})$, $n_{ii} = 1$. Let us define the Euler obstruction $\text{Eu}_{Y,0} \omega$ to be equal to 1 for a zero-dimensional variety Y (in particular $\text{Eu}_{\overline{V_0},0} \omega = 1$, $\text{Eu}_{N_{ii},0} \omega = 1$). In [EG05a], the following statement was proved.

Theorem 8. *One has*

$$\text{ind}_{\text{rad}}(\omega; V, 0) = \sum_{i=0}^q n_i \cdot \text{Eu}_{\overline{V_i},0} \omega.$$

To write an "inverse" of the formula of Theorem 8, suppose that the variety V is irreducible and $V = \overline{V_q}$. (Otherwise one can permit V to be reducible, but also permit the open stratum V_q to be not connected and dense; this does not change anything in Theorem 8.) Let m_{ij} be the (Möbius) inverse of the function n_{ij} on the partially ordered set of strata, i.e.

$$\sum_{i \preceq j \preceq k} n_{ij} m_{jk} = \delta_{ik}.$$

For $i \prec j$ one has

$$\begin{aligned} m_{ij} &= \sum_{i=k_0 \prec k_1 \prec \dots \prec k_r=j} (-1)^r n_{k_0 k_1} n_{k_1 k_2} \dots n_{k_{r-1} k_r} \\ &= (-1)^{\dim V - \dim V_i} \sum_{i=k_0 \prec \dots \prec k_r=j} \overline{\chi}(M_{\ell|N_{k_0 k_1}}) \dots \overline{\chi}(M_{\ell|N_{k_{r-1} k_r}}). \end{aligned}$$

Corollary 1. *One has*

$$\text{Eu}_{V,0} \omega = \sum_{i=0}^q m_{iq} \cdot \text{ind}_{\text{rad}}(\omega; \overline{V_i}, 0).$$

In particular

$$\text{Eu}_{V,0} df = (-1)^{\dim V - 1} \times \left(\overline{\chi}(M_{f|V}) + \sum_{i=0}^{q-1} \overline{\chi}(M_{f|\overline{V_i}}) \sum_{i=k_0 \prec \dots \prec k_r=q} \overline{\chi}(M_{\ell|N_{k_0 k_1}}) \dots \overline{\chi}(M_{\ell|N_{k_{r-1} k_r}}) \right).$$

In [LT81] the local Euler obstruction of a germ of a variety is related to polar invariants of the germ. A global version of the Euler obstruction of a variety and its connections with global polar invariants are discussed in [STV05a].

6. ALGEBRAIC, ANALYTIC, AND TOPOLOGICAL FORMULAE FOR INDICES

In the introduction, there were mentioned algebraic formulae for the index of an analytic vector field or a 1-form on a smooth manifold: as the

dimension of a ring in the complex setting and as the signature of a quadratic form in the real one. It is natural to try to look for analogues of such formulae for vector fields or 1-forms on singular varieties. This appeared to be a rather complicated problem. Other sorts of formulae: analytic (usually as certain residues) or topological ones are of interest as well.

A substantial progress in this direction was achieved for the GSV index.

Let $(V, 0)$ be the germ of a hypersurface in $(\mathbb{C}^{n+1}, 0)$ defined by a germ of a holomorphic function $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated singular point at 0. Let

$$X = \sum_{i=1}^{n+1} X_i \frac{\partial}{\partial x_i}$$

be a holomorphic vector field on \mathbb{C}^{n+1} tangent to V with an isolated (in $\mathbb{C}^{n+1}, 0$) zero at the origin. This implies that $Xf = hf$ for some $h \in \mathcal{O}_{\mathbb{C}^{n+1}, 0}$. Define the following ideals in $\mathcal{O}_{\mathbb{C}^{n+1}, 0}$:

$$\begin{aligned} J_f &:= (f_1, \dots, f_{n+1}), \\ J_X &:= (X_1, \dots, X_{n+1}), \\ J_1 &:= (h, J_X), \\ J_2 &:= (f, J_f), \\ J_3 &:= (f, J_X). \end{aligned}$$

X. Gómez-Mont [Góm98] has proved the following formula for the GSV index:

Theorem 9. *For a vector field X with an isolated zero in the ambient space tangent to the germ of a hypersurface $(V, 0)$ with an isolated singularity at 0, we have: for even n*

$$\text{Ind}_{\text{GSV}}(X; V, 0) = \dim \mathcal{O}_{\mathbb{C}^{n+1}, 0} / J_X + \dim \mathcal{O}_{\mathbb{C}^{n+1}, 0} / J_1 + \dim \mathcal{O}_{\mathbb{C}^{n+1}, 0} / J_2,$$

for n odd

$$\text{Ind}_{\text{GSV}}(X; V, 0) = \dim \mathcal{O}_{\mathbb{C}^{n+1}, 0} / J_2 + \dim \mathcal{O}_{\mathbb{C}^{n+1}, 0} / J_3.$$

In this case the GSV index coincides with the homological index. O. Klehn [Kl03] generalized this formula for the homological index to the case when the vector field has an isolated zero on the hypersurface singularity $(V, 0)$ but not necessarily in $(\mathbb{C}^N, 0)$. Recently, H.-Ch. Graf von Bothmer, X. Gómez-Mont and the first author gave formulae to compute the homological index in the case when V is a complete intersection [BEG05].

In [GM97, GM99] algebraic formulae for the index of a real analytic vector field with an algebraically isolated singular point at the origin tangent to a real hypersurface with an algebraically isolated singularity at 0 are derived. The index is expressed as the signature of a certain non-degenerate quadratic form for an even-dimensional hypersurface and as the difference of the signatures of two non-degenerate quadratic forms in the odd-dimensional case.

O. Klehn [Kl05] proved that the GSV index coincides with the dimension of a certain explicitly constructed vector space in the case when $(V, 0)$ is an isolated complete intersection singularity of dimension 1 and X is a vector field tangent to V with an isolated zero on V which is deformable in a certain way. He also obtained a signature formula for the real GSV index in the corresponding real analytic case generalizing the Eisenbud-Levine-Khimshiashvili formula.

Let $(V, 0) \subset (\mathbb{C}^{n+k}, 0)$ be an isolated complete intersection singularity defined by an analytic map

$$f = (f_1, \dots, f_k) : (\mathbb{C}^{n+k}, 0) \rightarrow (\mathbb{C}^k, 0).$$

Let X be a holomorphic vector field on \mathbb{C}^{n+k} defined in a suitable neighbourhood U of the origin, tangent to V , with an isolated singular point at the origin. Let C be the $k \times k$ matrix whose entries are holomorphic functions on U such that $X \cdot f = Cf$. Assume that (x_1, \dots, x_{n+k}) is a system of coordinates on U such that when X is written as

$$X = \sum_{i=1}^{n+k} X_i \frac{\partial}{\partial x_i},$$

the sequence $(X_1, \dots, X_n, f_1, \dots, f_k)$ is regular. Let J denote the Jacobian matrix

$$J = \left(\frac{\partial X_i}{\partial x_j} \right).$$

We denote by c_q the coefficient at t^q in the formal power series expansion of

$$\det \left(I_{n+k} - t \frac{\sqrt{-1}}{2\pi} J \right) \left[\det \left(I_k - t \frac{\sqrt{-1}}{2\pi} C \right) \right]^{-1}$$

in t , where I_{n+k} and I_k denote the identity matrices of sizes $n+k$ and k respectively. In [LSS95] the following formula for the GSV index is proved.

Theorem 10. *Let ε be small enough such that the real hypersurfaces $|X_i| = \varepsilon$ are in general position and let Z be the set $\{f = 0, |X_i| = \varepsilon, 1 \leq i \leq n\}$. We assume that Z is oriented so that*

$$d(\arg X_1) \wedge d(\arg X_2) \wedge \dots \wedge d(\arg X_n)$$

is positive. Then

$$\text{Ind}_{\text{GSV}}(X; V, 0) = \int_Z \frac{c_n dx_1 \wedge \dots \wedge dx_n}{\prod_{i=1}^n X_i}.$$

For generalizations of Theorem 10 and related results see [LS95, Su95, Su98, Su02, IS03].

Now let ω be a holomorphic 1-form on the ICIS $(V, 0) = f^{-1}(0) \subset (\mathbb{C}^{n+k}, 0)$, $f = (f_1, \dots, f_k) : (\mathbb{C}^{n+k}, 0) \rightarrow (\mathbb{C}^k, 0)$, i.e. the restriction to

$(V, 0)$ of a holomorphic 1-form

$$\omega = \sum_{i=1}^{n+k} A_i(x) dx_i$$

on $(\mathbb{C}^{n+k}, 0)$. Assume that ω has an isolated singular point at the origin (on $(V, 0)$). Let I is the ideal generated by f_1, \dots, f_k and the $(k+1) \times (k+1)$ -minors of the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n+k}} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_{n+k}} \\ A_1 & \cdots & A_{n+k} \end{pmatrix}.$$

Then one has the following formula for the GSV index [EG01, EG03a, EG03b].

Theorem 11. *One has*

$$\text{ind}_{\text{GSV}}(\omega; V, 0) = \dim \mathcal{O}_{\mathbb{C}^{n+k}, 0} / I.$$

(Note that there is a minor mistake in the proof of this theorem in [EG03a] which is corrected in [EG05d].) This formula was obtained by Lê D.T. and G.-M. Greuel for the case when ω is the differential of a function ([Gr75, Lê74]).

T. Gaffney [Ga05] described connections between the GSV index of a holomorphic 1-form on an ICIS and the multiplicity of pairs of certain modules.

In [EG05c], there was constructed a quadratic form on the algebra $\mathcal{A} := \mathcal{O}_{\mathbb{C}^{n+k}, 0} / I$ generalizing the Eisenbud-Levine-Khimshiashvili quadratic form defined for a smooth V . This is defined as follows.

Let $F : (\mathbb{C}^{n+k} \times \mathbb{C}_\varepsilon^M, 0) \rightarrow (\mathbb{C}^k \times \mathbb{C}_\varepsilon^M, 0)$ be a deformation of the map $f : (\mathbb{C}^{n+k}, 0) \rightarrow (\mathbb{C}^k, 0)$ ($F(x, \varepsilon) = (f_\varepsilon(x), \varepsilon)$, $f_0 = f$, $f_\varepsilon = (f_{1\varepsilon}, \dots, f_{k\varepsilon})$) and let ω_ε be a deformation of the form ω (defined in a neighbourhood of the origin in $\mathbb{C}^{n+k} \times \mathbb{C}_\varepsilon^M$) such that, for generic $\varepsilon \in (\mathbb{C}_\varepsilon^M, 0)$, the preimage $f_\varepsilon^{-1}(0)$ is smooth and (the restriction of) the form ω_ε to it has only non degenerate singular points.

Let $\Sigma \subset (\mathbb{C}_\varepsilon^M, 0)$ be the germ of the set of the values of the parameters ε from $(\mathbb{C}_\varepsilon^M, 0)$ such that either the preimage $f_\varepsilon^{-1}(0)$ is singular or the restriction of the 1-form ω_ε to it has degenerate singular points.

Let $\sum_{i=1}^n A_i dy^i$ be a 1-form on a smooth complex analytic manifold of dimension n (in local coordinates y^1, \dots, y^n). At points P where the 1-form ω vanishes, the Jacobian matrix $\mathcal{J} = \left(\frac{\partial A_i}{\partial y^j} \right)$ defines the tensor $\sum_{i,j} \frac{\partial A_i}{\partial y^j} dy^i \otimes dy^j$ of type $(0, 2)$ (i.e. a bilinear form on the tangent space). The determinant J of the matrix $\mathcal{J} = \left(\frac{\partial A_i}{\partial y^j} \right)$ is not a scalar (it depends on the choice of local

coordinates). Under a change of coordinates it is multiplied by the square of the Jacobian of the change of coordinates (since the Jacobian matrix \mathcal{J} is transformed to $C^T \mathcal{J} C$). Therefore it should be considered as the coefficient in the tensor

$$J(dy^1 \wedge \dots \wedge dy^m)^{\otimes 2}$$

of type $(0, 2m)$. In this sense the Jacobian of a 1-form is a sort of a "quadratic differential". To get a number $\tilde{J}(P)$, one can divide this tensor by the tensor square of a volume form.

Let us fix volume forms on \mathbb{C}^{n+k} and \mathbb{C}^k , say, the standard ones $\sigma_{n+k} = dx_1 \wedge \dots \wedge dx_{n+k}$ and $\sigma_k = dz_1 \wedge \dots \wedge dz_k$ where x_1, \dots, x_{n+k} and z_1, \dots, z_k are Cartesian coordinates in \mathbb{C}^{n+k} and in \mathbb{C}^k respectively. There exists (at least locally) an n -form σ on \mathbb{C}^n such that $\sigma_{n+k} = f^* \sigma_k \wedge \sigma$. Let $\varepsilon \notin \Sigma$. The restriction of the form σ to the manifold $f_\varepsilon^{-1}(0)$ is well defined and is a volume form on $f_\varepsilon^{-1}(0)$. We also denote it by σ . Let P_1, \dots, P_ν be the (non-degenerate) singular points of the 1-form ω_ε on the n -dimensional manifold $f_\varepsilon^{-1}(0)$. For $\varepsilon \notin \Sigma$ and a germ $\varphi \in \mathcal{O}_{\mathbb{C}^n, 0}$, let

$$(4) \quad R(\varphi, \varepsilon) := \sum_{i=1}^{\nu} \frac{\varphi(P_i)}{\tilde{J}_\varepsilon(P_i)}.$$

For a fixed φ the function $R_\varphi(\varepsilon) := R(\varphi, \varepsilon)$ is holomorphic in the complement of the bifurcation diagram Σ .

In [EG05c], it is proved that the function $R_\varphi(\varepsilon)$ has removable singularities on the bifurcation diagram Σ . This means that $R_\varphi(\varepsilon)$ can be extended to a holomorphic function on $(\mathbb{C}_\varepsilon^M, 0)$ (which we denote by the same symbol). Let

$$(5) \quad R(\varphi) := R_\varphi(0).$$

This defines a linear function R on $\mathcal{O}_{\mathbb{C}^n, 0}$. One can show that this linear function vanishes on the ideal I and hence defines a linear function R on the algebra \mathcal{A} . Using this function, we can define a quadratic form

$$Q(\varphi, \psi) = R(\varphi \cdot \psi)$$

on the algebra \mathcal{A} generalizing the Eisenbud-Levine-Khimshiashvili quadratic form defined for a smooth V .

A similar form can be defined on a module of differential forms associated to the pair (V, ω) , see [EG05c]. A special case of such a quadratic form (for curve singularities) was considered in [MvS90]. In the smooth case this quadratic form can be identified with the one on the algebra \mathcal{A} by an isomorphism of the underlying spaces. This is no longer true for an ICIS. In general, these two quadratic forms have even different ranks.

Let $g : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a real analytic function (i.e. a function taking real values on $\mathbb{R}^{n+1} \subset \mathbb{C}^{n+1}$) with an isolated singular point at 0. Then there is the following topological formula for the index of the gradient vector field $\text{grad } g$ at the origin. Let η be a sufficiently small positive real number. The complex conjugation induces involutions on the Milnor fibres

$M_{g,\eta}$ and $M_{g,-\eta}$ of g . Their actions on the homology groups $H_n(M_{g,\eta})$ and $H_n(M_{g,-\eta})$ will be denoted by σ_+ and σ_- respectively. On these homology groups we have the intersection forms $\langle \cdot, \cdot \rangle$. We consider the symmetric bilinear forms $Q_+(a, b) := \langle \sigma_+ a, b \rangle$ and $Q_-(a, b) := \langle \sigma_- a, b \rangle$ on $H_n(M_{g,\eta})$ and $H_n(M_{g,-\eta})$ respectively. The following statement was conjectured by Arnold [Ar78] and proved in [Gu84, Va85].

Theorem 12. *If n is even then*

$$\text{ind}_0 \text{grad } g = (-1)^{\frac{n}{2}} \frac{1}{2} (\text{sgn } Q_- - \text{sgn } Q_+),$$

where $\text{grad } g$ is the gradient vector field of the function g on \mathbb{R}^{n+1} .

If $\text{Var} : H_n(M_{g,\eta}, \partial M_{g,\eta}) \rightarrow H_n(M_{g,\eta})$ denotes the variation operator of the singularity of g , then there is another formula for the index in [Gu84]:

Theorem 13. *One has*

$$\text{ind}_0 \text{grad } g = (-1)^{\frac{n(n-1)}{2}} \text{sgn } \text{Var}^{-1} \sigma_+.$$

In [EG99] a generalization of such a formula for the radial index of a gradient vector field on an algebraically isolated real analytic isolated complete intersection singularity in \mathbb{C}^{n+k} was obtained.

In [Se95, SS96] formulas are given evaluating the GSV index of a singular point of a vector field on an isolated hypersurface or complete intersection singularity on a resolution of the singularity.

A. Esterov has extended the method of A.Khovanski of computation of intersection numbers from complete intersections to determinantal varieties. Using this he gave formulae for the GSV index of a 1-form on an ICIS and for its generalization to collections of 1-forms (see Section 8) in terms of Newton diagrams of the equations of the ICIS and of the components of the 1-forms. See [Es05b] for a short account of the results (a full version has been submitted to a journal; see also [Es05a] for some preliminary results).

In [Kl02] a residue formula for the GSV index of a holomorphic 1-form on an isolated surface singularity in the spirit of Theorem 10 is proved.

7. INDICES OF MEROMORPHIC 1-FORMS

Nonzero holomorphic vector fields (as well as holomorphic 1-forms) on a compact complex manifold rarely exist. Therefore it is interesting to consider meromorphic vector fields or 1-forms of which there are a lot (at least for projective manifolds). For a meromorphic vector field defined by a holomorphic section of the vector bundle $TM \otimes L$ (L is a holomorphic line bundle on M), the sum of the indices of its zeros is equal to the corresponding characteristic number $\langle c_n(TM \otimes L), [M] \rangle$ of the vector bundle $TM \otimes L$ (see, e.g., [BB70]).

Let α be a meromorphic 1-form on a compact complex manifold M^n which means that α is a holomorphic 1-form outside of a positive divisor D and in a neighbourhood of each point of M the form α can be written as $\hat{\alpha}/F$

where $F = 0$ is a local equation of the divisor D and $\hat{\alpha}$ is a holomorphic 1-form. Let L be the line bundle associated to the divisor D , i.e., L has a holomorphic section s with zeros on D . Then $\omega = s\alpha$ is a holomorphic section of the vector bundle $T^*M \otimes L$. In some constructions (say, as in [BB70]) one defines a meromorphic 1-form on M simply as a holomorphic section of the tensor product $T^*M \otimes L$ for a holomorphic line bundle L . This definition is somewhat different from the one formulated above. E.g., in this case only the class of the divisor of poles of a meromorphic 1-form is defined, not the divisor itself. Moreover, in this setting, the value of a meromorphic 1-form on a vector field is not a function, but a section of the line bundle L . In the sequel we use the notation ω for a section of the vector bundle $T^*M \otimes L$, for short calling it a meromorphic 1-form as well. Suppose that the section ω has isolated zeros. Then the sum of their indices is equal to the characteristic number $\langle c_n(T^*M \otimes L), [M] \rangle$ and thus depends on L .

For a meromorphic 1-form on a smooth compact complex curve M the characteristic number $\langle c_1(T^*M), [M] \rangle = -\chi(M)$ is equal to the number of zeros minus the number of poles counted with multiplicities. Therefore to express the Euler characteristic of a compact manifold in terms of singularities of a meromorphic 1-form one has to take the divisor D of its poles into account as well. If a meromorphic 1-form on a manifold M^n is defined simply as a section of $T^*M \otimes L$, the divisor of poles is not defined and thus one also cannot define singular points of the 1-form on its pole locus.

One would like to give a Poincaré-Hopf type formula for meromorphic 1-forms, i.e., to express the Euler characteristic of a compact manifold or of a smoothing of a complete intersection in terms of singularities of a meromorphic 1-form. For that it is possible to introduce a suitable notion of an index of a germ of a meromorphic 1-form (with an additional structure) on an ICIS, so that the indices of the singular points sum up to (plus-minus) the Euler characteristic of a smoothing. There are several descriptions of the index of a meromorphic 1-form, one of them being the alternating sum of the dimensions of certain algebras.

A Poincaré-Hopf type formula can be considered as one describing a localization of an invariant (say, of the Euler characteristic) of a manifold at singular points of, say, a vector field or a 1-form, i.e., a representation of the invariant as the sum of integer invariants ("indices") corresponding to singular points. Therefore one first has to define singular points. Let M^n be a complex manifold, let L be a line bundle on M , and let ω be a holomorphic section of the bundle $T^*M \otimes L$. One has zeros of ω on M , but the pole locus of ω is not well defined. In order to discuss singular points of ω on its pole locus (which is necessary as we saw in the example when M was a curve), we have to fix this locus. This means that we have to choose a holomorphic section $s = s_1$ of the line bundle L or to choose its zero divisor $D = D_1$. One can say that we have to consider $\alpha = \omega/s$, i.e., to proceed with our initial definition of a meromorphic 1-form.

For the further setting we suppose that the divisor D of poles of the 1-form ω is non-singular (in particular, reduced). This is not very essential for this section, however, this makes the discussion simpler. Since D is a submanifold of M , there is a well defined map $T^*M|_D \rightarrow T^*D$ and thus $(T^*M \otimes L)|_D \rightarrow T^*D \otimes L|_D$ (the restriction of meromorphic 1-forms to D). (Using the 1-form α , one can also define only a section of the vector bundle $T^*D \otimes L|_D$, but not a meromorphic 1-form on D with precisely defined pole locus.)

Let ω_1 be the restriction of ω to D_1 . It is a holomorphic section of the vector bundle $T^*D \otimes L|_D$. Its zeros are well defined and should be considered as singular points of the meromorphic 1-form ω on the pole locus. To discuss its singular points on its pole locus we again have to fix a divisor. Suppose that there exists a (positive) divisor D_2 on M which is the zero locus of another section s_2 of the same line bundle L and which intersects D_1 transversally (in particular, this means that D_2 is non-singular at its intersection points with D_1 and $D_1 \cap D_2$ is non-singular as well). One has the restriction of ω to $D_1 \cap D_2$ and its zeros there.

Going on this way we arrive at the situation when we have fixed n divisors D_1, \dots, D_n (zeros of sections s_1, \dots, s_n of the line bundle L) so that, for each $i = 1, \dots, n$, $D_1 \cap \dots \cap D_i$ is non-singular. The set of singular points of the 1-form ω is the union of the zeros of ω itself and of the restrictions of ω to $D_1 \cap \dots \cap D_i$ for all $i = 1, \dots, n$. (For $i = n$ the intersection $D_1 \cap \dots \cap D_n$ is zero-dimensional and all its points should be considered as zeros of the 1-form ω .) One can say that we have to consider a collection of meromorphic 1-forms $\omega/s_1, \dots, \omega/s_n$ on M proportional to each other.

There is the following Poincaré-Hopf type formula for meromorphic 1-forms. Let M^n be a compact complex manifold, let ω be a meromorphic 1-form on M , that is, a holomorphic section of the bundle $T^*M \otimes L$ where L is a holomorphic line bundle with nonzero holomorphic sections. Suppose that $D_1 = D, D_2, \dots, D_n$ are zero divisors of holomorphic sections of the line bundle L such that, for each $i = 1, \dots, n$, $D_1 \cap \dots \cap D_i$ is non-singular. Suppose that the form ω itself and its restrictions to the submanifolds $D_1 \cap \dots \cap D_i$, $i = 1, \dots, n$, have only isolated zeros. Let m_0 (respectively m_i , $i = 1, \dots, n$) be the number of zeros of the form ω (respectively, of the restriction of ω to the intersection $D_1 \cap \dots \cap D_i$) counted with multiplicities. In particular m_n is the number of points in $D_1 \cap \dots \cap D_n$.

Theorem 14. $c_n(T^*M)[M] = (-1)^n \chi(M^n) = m_0 - m_1 + \dots + (-1)^n m_n$.

Remark. One can show that

$$m_0 = (-1)^n (\chi(M) - \chi(D)) = (-1)^n \chi(M \setminus D).$$

Suppose that all zeros of ω are outside of D (one can consider this situation as the generic one). Let s be the holomorphic section of the line bundle L with zeros on D . Then $\alpha = \omega/s$ is a holomorphic 1-form on $M \setminus D$ with simple poles along D . So in this case the number m_0 of zeros of the

holomorphic 1-form α on $M \setminus D$ coincides with $(-1)^n \chi(M \setminus D)$. This is the relation which holds for holomorphic 1-forms on compact manifolds ($M \setminus D$ is not compact).

Example 4. Let

$$\alpha = \frac{xdy - ydx + dz}{x^2 + 4y^2 + z^2 + 1}$$

be a meromorphic 1-form on the projective space \mathbb{CP}^3 (x, y, z are affine coordinates). One can see that the zeros of the corresponding ω on \mathbb{CP}^3 and also the zeros of $\omega|_D$, $D = D_1 = \{x^2 + 4y^2 + z^2 + 1 = 0\}$, are isolated and $m_0 = 0$, $m_1 = 4$. To define other singular points one has to choose D_2 and D_3 (e.g., $D_2 = \{x^2 + y^2 + 4z^2 = 0\}$, $D_3 = \{x^2 + y^2 + z^2 = 0\}$). One has $m_2 = 8$, $m_3 = 8$, $0 - 4 + 8 - 8 = -4 = (-1)^3 \chi(\mathbb{CP}^3)$. Note that as a meromorphic 1-form on \mathbb{CP}^3 with poles on the hypersurface $P_d(x, y, z) = 0$, $\deg P_d = d$, it is natural to take

$$\frac{A_{d-2}dx + B_{d-2}dy + C_{d-2}dz}{P_d},$$

where A_{d-2} , B_{d-2} , and C_{d-2} are polynomials in x, y, z of degree $d-2$ (α is not of this form). However, one can show that such a 1-form has non-isolated zeros on \mathbb{CP}^3 (at infinity).

Now let V^n be a compact subvariety of a complex manifold M^{n+k} such that in a neighbourhood of each point V is a complete intersection in M with only isolated singularities (M is not supposed to be compact). Let ω be a meromorphic 1-form on M , i.e., a holomorphic section of the bundle $T^*M \otimes L$, where L is a line bundle on M with nonzero holomorphic sections. By a smoothing \tilde{V} of V we understand a smooth (C^∞) manifold which is obtained from V by smoothing its singular points (which are ICIS) in the usual way (as complex ICIS). (If V is a complete intersection in the projective space \mathbb{CP}^{n+k} , a smoothing of V can be obtained as an analytic submanifold of \mathbb{CP}^{n+k} as well.)

It is possible to formulate a Poincaré-Hopf type formula for the Euler characteristic of V and of its smoothing \tilde{V} in terms of singular points of the 1-form ω on V . As above we suppose that there exist n holomorphic sections s_1, \dots, s_n of the line bundle L with zero divisors D_1, \dots, D_n such that, for each $i = 1, \dots, n$, the intersection $V \cap D_1 \cap \dots \cap D_i$ is of dimension $n - i$ and has only isolated singularities (and thus is reduced for $i < n$). We also suppose that the restriction of ω to the non-singular part of V and its restriction to the non-singular part of $V \cap D_1 \cap \dots \cap D_i$ ($i = 1, \dots, n$) have only isolated zeros. As singular points of the 1-form ω on V we consider all zeros of its restrictions to the non-singular parts of V and of the intersections $V \cap D_1 \cap \dots \cap D_i$ ($i = 1, \dots, n$) and all singular points of V and of $V \cap D_1 \cap \dots \cap D_i$ as well.

Let P be a singular point of the 1-form ω on V . Suppose that the divisors D_1, \dots, D_ℓ ($0 \leq \ell \leq n$) pass through the point P and (if $\ell < n$) $D_{\ell+1}$ does

not. In a neighbourhood of the point P , in some local coordinates on M centred at the point P , we have the following situation. The variety (ICIS) V (possibly non-singular) is defined by k equations $f_1 = \dots = f_k = 0$. Let divisors D_1, \dots, D_ℓ be defined by equations $f_{k+1} = 0, \dots, f_{k+\ell} = 0$. Let us remind that $\{f_1 = \dots = f_k = f_{k+1} = \dots = f_{k+i} = 0\}$ is an ICIS for each $i = 0, 1, \dots, \ell$. After choosing a local trivialization of the line bundle L , the 1-form ω can be written as $\sum_{i=1}^{n+k} A_i(x) dx_i$ where $A_i(x)$ are holomorphic. One can say that we consider a collection of meromorphic 1-forms $\omega/f_{k+1}, \dots, \omega/f_{k+\ell}$ on V proportional to each other. Let us denote the set of local data $(V, \omega, f_{k+1}, \dots, f_{k+\ell})$ by Ω .

Let $\text{ind}^{(0)}\Omega$ (respectively $\text{ind}^{(i)}\Omega$, $i = 1, \dots, \ell$) be the index of the holomorphic 1-form ω on V (respectively on the ICIS $V \cap D_1 \cap \dots \cap D_i$) defined in Section 3.

Definition: The alternating sum

$$\text{ind}_P\Omega = \text{ind}^{(0)}\Omega - \text{ind}^{(1)}\Omega + \dots + (-1)^\ell \text{ind}^{(\ell)}\Omega$$

is called the index of the 1-form ω at the point P with respect to the divisors D_1, \dots, D_n .

According to the statements from Section 3 (see also Section 6 for the last point) one has the following three equivalent descriptions of the index $\text{ind}_P\Omega$.

1. Let B_δ be the ball of sufficiently small radius δ centred at the origin in \mathbb{C}^{n+k} and let S_δ be its boundary. Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k, \dots, \varepsilon_{k+\ell}) \in (\mathbb{C}^{k+\ell}, 0)$ be small enough and such that $\varepsilon^{(k+i)} = (\varepsilon_1, \dots, \varepsilon_k, \dots, \varepsilon_{k+i})$ is not a critical value of the map $F_{k+i} = (f_1, \dots, f_k, \dots, f_{k+i}) : (\mathbb{C}^{n+k}, 0) \rightarrow (\mathbb{C}^{k+i}, 0)$ for each $i = 0, 1, \dots, \ell$. The restriction of the 1-form ω to the smooth manifold $V_i = F_{k+i}^{-1}(\varepsilon^{(k+i)})$ has isolated zeros in B_δ . Let m_i be the number of them counted with multiplicities. Then $\text{ind}^{(i)}\Omega = m_i$, $\text{ind}_P\Omega = m_0 - m_1 + \dots + (-1)^\ell m_\ell$.

2. Let K_i ($i = 0, 1, \dots, \ell$) be the link of the ICIS $V \cap D_1 \cap \dots \cap D_i$, i.e., the intersection $V \cap D_1 \cap \dots \cap D_i \cap S_\delta$. Let d_i be the degree of the map

$$(\omega, df_1, \dots, df_k, \dots, df_{k+i}) : K_i \rightarrow W_{k+i+1}(\mathbb{C}^{n+k})$$

($W_{k+i+1}(\mathbb{C}^{n+k})$ is the Stiefel manifold of $(k+i+1)$ -frames in the dual \mathbb{C}^{n+k}). Then $\text{ind}^{(i)}\Omega = d_i$, $\text{ind}_P\Omega = d_0 - d_1 + \dots + (-1)^\ell d_\ell$.

3. Let I_i ($i = 0, 1, \dots, \ell$) be the ideal of the ring $\mathcal{O}_{\mathbb{C}^{n+k}, 0}$ of germs of holomorphic functions of $n+k$ variables at the origin generated by $f_1, \dots, f_k, \dots, f_{k+i}$ and the $(k+i+1) \times (k+i+1)$ -minors of the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_{n+k}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_{k+i}}{\partial x_1} & \dots & \frac{\partial f_{k+i}}{\partial x_{n+k}} \\ A_1 & \dots & A_{n+k} \end{pmatrix}.$$

Let $\nu_i = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^{n+k},0}/I_i$. Then $\text{ind}^{(i)}\Omega = \nu_i$, $\text{ind}_P\omega = \nu_0 - \nu_1 + \dots + (-1)^\ell \nu_\ell$.
One has the following Poincaré–Hopf type formula.

Theorem 15.

$$\sum_P \text{ind}_P\Omega = (-1)^n \chi(\tilde{V})$$

where \tilde{V} is a smoothing of the variety V .

In order to get the Euler characteristic of the variety V itself one has to correct this formula by taking the Milnor numbers of the singular points of V into account. For $P \in V$, let μ_P denote the Milnor number of the ICIS V at the point P (see, e.g., [Lo84]). Note that, if V is non-singular at P , then $\mu_P = 0$.

Theorem 16.

$$(-1)^n \chi(V) = \sum_P (\text{ind}_P\Omega - \mu_P).$$

Remark. The Milnor number μ_P can also be written as an alternating sum of indices of holomorphic 1-forms on several ICIS. One can say that the term μ_P corresponds to the difference between two possible definitions of the index as in [SS98, Proposition 1.4].

In some cases it is natural to consider the situation when the pole locus of the meromorphic 1-form ω is a multiple of an irreducible divisor. Examples are:

- 1) A polynomial 1-form on \mathbb{C}^n is a meromorphic 1-form on \mathbb{CP}^n the pole locus of which is a multiple of the infinite hyperplane \mathbb{CP}_∞^{n-1} .
- 2) Let f be a meromorphic function with the pole locus kD where D is an irreducible divisor (i.e., in a neighbourhood of each point f can be written as \tilde{f}/F^k where \tilde{f} is holomorphic and $F = 0$ is a local equation of D). Then its differential df is a meromorphic 1-form with the pole locus $(k+1)D$ (i.e., in a neighbourhood of each point df can be written as $\tilde{\omega}/F^{k+1}$ where $\tilde{\omega}$ is a holomorphic 1-form). A polynomial function of degree k on \mathbb{C}^n is a meromorphic function on \mathbb{CP}^n with the pole locus $k\mathbb{CP}_\infty^{n-1}$.

Let us give a version of Theorem 14 for this case. The fact that the pole divisor of a 1-form is multiple means that the corresponding line bundle is a power of another one. Let ω be a meromorphic 1-form on a compact complex manifold M^n , that is, a holomorphic section of the bundle $T^*M \otimes L$, where $L = \lambda^k$, $k > 1$. Suppose that $D_1 = D$, D_2, \dots, D_n are zero divisors of holomorphic sections of the line bundle λ such that, for each $i = 1, \dots, n$, $D_1 \cap \dots \cap D_i$ is non-singular and the form ω itself and its restrictions to $D_1 \cap \dots \cap D_i$ have isolated zeros. Let m_0 (respectively m_i , $i = 1, \dots, n$) be the number of zeros of the 1-form ω (respectively of its restriction to $D_1 \cap \dots \cap D_i$) counted with multiplicities.

Theorem 17.

$$\langle c_n(T^*M), [M] \rangle = (-1)^n \chi(M) = m_0 - km_1 + \dots + (-1)^n km_n.$$

Remark. In [HS98, Theorem 3.1] there is given a formula for the sum of the residues corresponding to the singular points of the foliation on the projective plane \mathbb{CP}^2 given by $df = 0$, where f is a polynomial function of degree k on \mathbb{C}^2 (which defines a meromorphic function on \mathbb{CP}^2). In our terms this is the formula for the number m_0 of zeros of the meromorphic 1-form df , the pole locus of which is $k + 1$ times the infinite line. Thus Theorem 17 can be considered as a generalization of [HS98, Theorem 3.1] to higher dimensions and to meromorphic 1-forms which are not, in general, differentials of functions.

8. INDICES OF COLLECTIONS OF 1-FORMS

One can say that all the indices discussed above are connected with the Euler characteristic (e.g. through the Poincaré–Hopf type theorem). If M is a complex analytic manifold of dimension n , then its Euler characteristic $\chi(M)$ is the characteristic number $\langle c_n(TM), [M] \rangle = (-1)^n \langle c_n(T^*M), [M] \rangle$, where TM is the tangent bundle of the manifold M , T^*M is the dual bundle, and c_n is the corresponding Chern class. One can try to find generalizations of some notions and statements about indices of vector fields or of 1-forms for other characteristic numbers (different from $\langle c_n(TM), [M] \rangle$ or $\langle c_n(T^*M), [M] \rangle$). This was made in [EG05b] and [EG05d] for the GSV index and for the Euler obstruction.

The top Chern class of a vector bundle is the (first) obstruction to existence of a non-vanishing section. Other Chern classes are obstructions to existence of linear independent collections of sections. Therefore instead of 1-forms on a complex variety we consider collections of 1-forms.

Let $\pi : E \rightarrow M$ be a complex analytic vector bundle of rank m over a complex analytic manifold M of dimension n . It is known that the $(2(n - k))$ -dimensional cycle Poincaré dual to the characteristic class $c_k(E)$ ($k = 1, \dots, m$) is represented by the set of points of the manifold M where $m - k + 1$ generic sections of the vector bundle E are linearly dependent (cf., e.g., [GH78, p. 413]).

Let $\mathcal{M}(p, q)$, $D_{p,q}$, and $W_{p,q}$ be the spaces defined in Section 3. Let $\mathbf{k} = (k_1, \dots, k_s)$ be a sequence of positive integers with $\sum_{i=1}^s k_i = k$. Consider the space $\mathcal{M}_{m,\mathbf{k}} = \prod_{i=1}^s \mathcal{M}(m, m - k_i + 1)$ and the subvariety $D_{m,\mathbf{k}} = \prod_{i=1}^s D_{m, m - k_i + 1}$ in it. The variety $D_{m,\mathbf{k}}$ consists of sets $\{A_i\}$ of $m \times (m - k_i + 1)$ matrices such that $\text{rk } A_i < m - k_i + 1$ for each $i = 1, \dots, s$. Since $D_{m,\mathbf{k}}$ is irreducible of codimension k , its complement $W_{m,\mathbf{k}} = \mathcal{M}_{m,\mathbf{k}} \setminus D_{m,\mathbf{k}}$ is $(2k - 2)$ -connected, $H_{2k-1}(W_{m,\mathbf{k}}) \cong \mathbb{Z}$, and there is a natural choice of a generator of the latter group. This choice defines a degree (an integer) of a map from an oriented manifold of dimension $2k - 1$ to the manifold $W_{m,\mathbf{k}}$.

Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be an n -dimensional isolated complete intersection singularity (ICIS) defined by equations $f_1 = \dots = f_{N-n} = 0$ ($f_i \in \mathcal{O}_{\mathbb{C}^N, 0}$). Let f be the analytic map $(f_1, \dots, f_{N-n}) : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^{N-n}, 0)$ ($V = f^{-1}(0)$). Let $\{X_j^{(i)}\}$ be a collection of vector fields on a neighbourhood of the origin in $(\mathbb{C}^N, 0)$ ($i = 1, \dots, s$; $j = 1, \dots, n - k_i + 1$; $\sum k_i = n$) which are tangent to the ICIS $(V, 0) = \{f_1 = \dots = f_{N-n} = 0\} \subset (\mathbb{C}^N, 0)$ at non-singular points of V . We say that a point $p \in V \setminus \{0\}$ is non-singular for the collection $\{X_j^{(i)}\}$ on V if at least for some i the vectors $X_1^{(i)}(p), \dots, X_{n-k_i+1}^{(i)}(p)$ are linearly independent. Suppose that the collection $\{X_j^{(i)}\}$ has no singular points on V outside of the origin in a neighbourhood of it. Let U be a neighbourhood of the origin in \mathbb{C}^N where all the functions f_r ($r = 1, \dots, N - n$) and the vector fields $X_j^{(i)}$ are defined and such that the collection $\{X_j^{(i)}\}$ has no singular points on $(V \cap U) \setminus \{0\}$. Let $S_\delta \subset U$ be a sufficiently small sphere around the origin which intersects V transversally and denote by $K = V \cap S_\delta$ the link of the ICIS $(V, 0)$. The manifold K has a natural orientation as the boundary of a complex analytic manifold. Let Ψ_V be the mapping from $V \cap U$ to $\mathcal{M}_{n, \mathbf{k}}$ which sends a point $x \in V \cap U$ to the collection of $N \times (N - k_i + 1)$ -matrices

$$\{(\text{grad } f_1(x), \dots, \text{grad } f_{N-n}(x), X_1^{(i)}(x), \dots, X_{n-k_i+1}^{(i)}(x))\}, \quad i = 1, \dots, s.$$

Here $\text{grad } f_r$ is the gradient vector field of f_r defined in Section 3. Its restriction ψ_V to the link K maps K to the subset $W_{N, \mathbf{k}}$.

Definition: The index $\text{ind}_{V, 0}\{X_j^{(i)}\}$ of the collection of vector fields $\{\omega_j^{(i)}\}$ on the ICIS V is the degree of the mapping $\psi_V : K \rightarrow W_{N, \mathbf{k}}$.

For $s = 1$, $k_1 = n$, this index is the GSV index of a vector field on an ICIS (see Section 3).

Let $V \subset \mathbb{C}P^N$ be an n -dimensional complete intersection with isolated singular points, $V = \{f_1 = \dots = f_{N-n} = 0\}$ where f_i are homogeneous functions in $(N+1)$ variables. Let $\{X_j^{(i)}\}$ be a collection of continuous vector fields on $\mathbb{C}P^N$ which are tangent to V . Let \tilde{V} be a smoothing of the complete intersection V , i.e. \tilde{V} is defined by $N - n$ equations $\tilde{f}_1 = \dots = \tilde{f}_{N-n} = 0$ where the homogeneous functions \tilde{f}_i are small perturbations of the functions f_i and \tilde{V} is smooth.

The usual description of Chern classes of a vector bundle as obstructions to existence of several linear independent sections of the bundle implies the following statement.

Theorem 18. *One has*

$$\sum_{p \in V} \text{ind}_p \{X_j^{(i)}\} = \left\langle \prod_{i=1}^s c_{k_i}(T\tilde{V}), [\tilde{V}] \right\rangle,$$

where \tilde{V} is a smoothing of the complete intersection V .

Now let $\{\omega_j^{(i)}\}$ be a collection of (continuous) 1-forms on a neighbourhood of the origin in $(\mathbb{C}^N, 0)$ with $i = 1, \dots, s$, $j = 1, \dots, n - k_i + 1$, $\sum k_i = n$. We say that a point $p \in V \setminus \{0\}$ is non-singular for the collection $\{\omega_j^{(i)}\}$ on V if at least for some i the restrictions of the 1-forms $\omega_j^{(i)}(p)$ to the tangent space $T_p V$ are linearly independent. Suppose that the collection $\{\omega_j^{(i)}\}$ has no singular points on V outside of the origin in a neighbourhood of it. As above let $K = V \cap S_\delta$ be the link of the ICIS $(V, 0)$ (all the functions f_r and the 1-forms $\omega_j^{(i)}$ are defined in a neighbourhood of the ball B_δ). Let Ψ_V be the mapping from $V \cap U$ to $\mathcal{M}_{n, \mathbf{k}}$ which sends a point $x \in V \cap U$ to the collection of $N \times (N - k_i + 1)$ -matrices

$$\{(df_1(x), \dots, df_{N-n}(x), \omega_1^{(i)}(x), \dots, \omega_{n-k_i+1}^{(i)}(x))\}, \quad i = 1, \dots, s.$$

Its restriction ψ_V to the link K maps K to the subset $W_{N, \mathbf{k}}$.

Definition: The *index* $\text{ind}_{V,0}\{\omega_j^{(i)}\}$ of the collection of 1-forms $\{\omega_j^{(i)}\}$ on the ICIS V is the degree of the mapping $\psi_V : K \rightarrow W_{N, \mathbf{k}}$.

One can easily see that the index $\text{ind}_{V,0}\{\omega_j^{(i)}\}$ is equal to the intersection number of the germ of the image of the mapping Ψ_V with the variety $D_{N, \mathbf{k}}$. If all the 1-forms $\omega_j^{(i)}$ are complex analytic, the mapping Ψ_V is complex analytic as well.

For $s = 1$, $k_1 = n$, this index is the GSV index of a 1-form (Section 3).

Let $V \subset \mathbb{CP}^N$ be an n -dimensional complete intersection with isolated singular points, $V = \{f_1 = \dots = f_{N-n} = 0\}$ where f_i are homogeneous functions in $(N + 1)$ variables. Let L be a complex line bundle on V and let $\{\omega_j^{(i)}\}$ be a collection of continuous 1-forms on V with values in L . Here this means that the forms $\omega_j^{(i)}$ are continuous sections of the vector bundle $T^*V \otimes L$ outside of the singular points of V . Since, in a neighbourhood of each point p , the vector bundle L is trivial, one can define the index $\text{ind}_p\{\omega_j^{(i)}\}$ of the collection of 1-forms $\{\omega_j^{(i)}\}$ at the point p just in the same way as in the local setting above. Let \tilde{V} be a smoothing of the complete intersection V . One can consider L as a line bundle on the smoothing \tilde{V} of the complete intersection V as well (e.g., using the pull back along a projection of \tilde{V} to V ; L is not, in general, complex analytic). The collection $\{\omega_j^{(i)}\}$ of 1-forms can also be extended to a neighbourhood of V in such a way that it will define a collection of 1-forms on the smoothing \tilde{V} (also denoted by $\{\omega_j^{(i)}\}$) with isolated singular points. The sum of the indices of the collection $\{\omega_j^{(i)}\}$ on the smoothing \tilde{V} of V in a neighbourhood of the point p is equal to the index $\text{ind}_{V,p}\{\omega_j^{(i)}\}$.

One has the following analogue of Proposition 4 for 1-forms.

Theorem 19. *One has*

$$\sum_{p \in V} \text{ind}_p \{\omega_j^{(i)}\} = \langle \prod_{i=1}^s c_{k_i}(T^* \tilde{V} \otimes L), [\tilde{V}] \rangle,$$

where \tilde{V} is a smoothing of the complete intersection V .

As above, let $(V, 0) \subset (\mathbb{C}^N, 0)$ be the ICIS defined by the equations $f_1 = \dots = f_{N-n} = 0$. Let $\{\omega_j^{(i)}\}$ ($i = 1, \dots, s; j = 1, \dots, n - k_i + 1$) be a collection of 1-forms on a neighbourhood of the origin in \mathbb{C}^N without singular points on $V \setminus \{0\}$ in a neighbourhood of the origin. If all the 1-forms $\omega_j^{(i)}$ are complex analytic, there exists an algebraic formula for the index $\text{ind}_{V,0} \{\omega_j^{(i)}\}$ of the collection $\{\omega_j^{(i)}\}$ similar to that from Theorem 11.

Let $I_{V, \{\omega_j^{(i)}\}}$ be the ideal in the ring $\mathcal{O}_{\mathbb{C}^N, 0}$ generated by the functions f_1, \dots, f_{N-n} and by the $(N - k_i + 1) \times (N - k_i + 1)$ minors of all the matrices

$$(df_1(x), \dots, df_{N-n}(x), \omega_1^{(i)}(x), \dots, \omega_{n-k_i+1}^{(i)}(x))$$

for all $i = 1, \dots, s$.

Theorem 20. (see [EG05b])

$$\text{ind}_{V,0} \{\omega_j^{(i)}\} = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^N, 0} / I_{V, \{\omega_j^{(i)}\}}.$$

Remark. A formula similar to that of Theorem 20 does not exist for collections of vector fields. A reason is that in this case the index is the intersection number with $D_{N, \mathbf{k}}$ of the image of the ICIS $(V, 0)$ under a map which is not complex analytic. Moreover, in some cases this index can be negative (see e.g. [GSV91, Proposition 2.2]).

Example 5. Let V be the (singular) quadric $x_1^2 + x_2^2 + x_3^2 = 0$ in the projective space \mathbb{CP}^3 with the coordinates $(x_0 : x_1 : x_2 : x_3)$. Let the 1-forms from the collection $\{\{\omega_1^{(1)}, \omega_2^{(1)}\}, \{\omega_1^{(2)}, \omega_2^{(2)}\}\}$ be defined in the affine chart $\mathbb{C}^3 = \{x_0 = 1\}$ of the projective space \mathbb{CP}^3 as $\omega_1^{(1)} = dx_1$, $\omega_2^{(1)} = x_2 dx_3 - x_3 dx_2$, $\omega_1^{(2)} = dx_2$, $\omega_2^{(2)} = x_1 dx_3 - x_3 dx_1$. They are holomorphic 1-forms on the projective space \mathbb{CP}^3 with values in the line bundle $\mathcal{O}(2)$. It is easy to see that being restricted to the quadric V the indicated collection of 1-forms has no singular points outside of the singular point $(1 : 0 : 0 : 0)$ of V itself. Theorem 19 says that the index of this point is equal to $\langle c_1^2(T^* \tilde{V} \otimes i^* \mathcal{O}(2)), [\tilde{V}] \rangle$, where \tilde{V} is a non-singular quadric, $i^* \mathcal{O}(2)$ is the restriction of the line bundle $\mathcal{O}(2)$ to \tilde{V} . Taking into account that $\tilde{V} \cong (\mathbb{CP}^1)^2$, one gets that this characteristic number is equal to 8. Now by Theorem 20 the index of the indicated collection of 1-forms on the quadric V at the origin in $\mathbb{C}^3 \subset \mathbb{CP}^3$ is equal to $\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3, 0} / \langle x_1^2 + x_2^2 + x_3^2, x_2^2 + x_3^2, x_1^2 + x_3^2 \rangle$ which is also equal to 8.

There exists a generalization of the notion of the Euler obstruction to collections of 1-forms corresponding to different Chern numbers.

Let $(V^n, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a purely n -dimensional reduced complex analytic variety at the origin (generally speaking with a non-isolated singularity). Let $\mathbf{k} = \{k_i\}$, $i = 1, \dots, s$, be a fixed partition of n (i.e., k_i are positive integers, $\sum_{i=1}^s k_i = n$). Let $\{\omega_j^{(i)}\}$ ($i = 1, \dots, s$, $j = 1, \dots, n - k_i + 1$) be a collection of germs of 1-forms on $(\mathbb{C}^N, 0)$ (not necessarily complex analytic; it suffices that the forms $\omega_j^{(i)}$ are complex linear functions on \mathbb{C}^N continuously depending on a point of \mathbb{C}^N). Let $\varepsilon > 0$ be small enough so that there is a representative V of the germ $(V, 0)$ and representatives $\omega_j^{(i)}$ of the germs of 1-forms inside the ball $B_\varepsilon(0) \subset \mathbb{C}^N$.

Definition: A point $P \in V$ is called a *special* point of the collection $\{\omega_j^{(i)}\}$ of 1-forms on the variety V if there exists a sequence $\{P_m\}$ of points from the non-singular part V_{reg} of the variety V such that the sequence $T_{P_m} V_{\text{reg}}$ of the tangent spaces at the points P_m has a limit L (in $G(n, N)$) and the restrictions of the 1-forms $\omega_1^{(i)}, \dots, \omega_{n-k_i+1}^{(i)}$ to the subspace $L \subset T_P \mathbb{C}^N$ are linearly dependent for each $i = 1, \dots, s$. The collection $\{\omega_j^{(i)}\}$ of 1-forms has an *isolated special point* on $(V, 0)$ if it has no special points on V in a punctured neighbourhood of the origin.

Remark. If the 1-forms $\omega_j^{(i)}$ are complex analytic, the property to have an isolated special point is a condition on the classes of these 1-forms in the module

$$\Omega_{V,0}^1 = \Omega_{\mathbb{C}^N,0}^1 / \{f \cdot \Omega_{\mathbb{C}^N,0}^1 + df \cdot \mathcal{O}_{\mathbb{C}^N,0} | f \in \mathcal{I}_V\}$$

of germs of 1-forms on the variety V (\mathcal{I}_V is the ideal of germs of functions vanishing on V).

Remark. For the case $s = 1$ (and therefore $k_1 = n$), i.e. for one 1-form ω , we discussed the notion of a *singular* point of the 1-form ω on V . One can easily see that a special point of the 1-form ω on V is singular, but not vice versa. (E.g. the origin is a singular point of the 1-form dx on the cone $\{x^2 + y^2 + z^2 = 0\}$, but not a special one.) On a smooth variety these two notions coincide.

Definition: A special (singular) point of a collection $\{\omega_j^{(i)}\}$ of germs of 1-forms on a smooth n -dimensional variety V is *non-degenerate* if the map $\Psi_V : V \cap U \rightarrow \mathcal{M}_{n,\mathbf{k}}$ described above is transversal to $D_{n,\mathbf{k}} \subset \mathcal{M}_{n,\mathbf{k}}$ at a non-singular point of it.

Let

$$\mathcal{L}_{\mathbf{k}} = \prod_{i=1}^s \prod_{j=1}^{n-k_i+1} \mathbb{C}_{ij}^{N*}$$

be the space of collections of linear functions on \mathbb{C}^N (i.e. of 1-forms with constant coefficients). The following statement holds.

Proposition 7. *There exists an open and dense set $U \subset \mathcal{L}_{\mathbf{k}}$ such that each collection $\{\ell_j^{(i)}\} \in U$ has only isolated special points on V and, moreover, all these points belong to the smooth part V_{reg} of the variety V and are non-degenerate.*

Corollary 2. *Let $\{\omega_j^{(i)}\}$ be a collection of 1-forms on V with an isolated special point at the origin. Then there exists a deformation $\{\tilde{\omega}_j^{(i)}\}$ of the collection $\{\omega_j^{(i)}\}$ whose special points lie in V_{reg} and are non-degenerate. Moreover, as such a deformation one can use $\{\omega_j^{(i)} + \lambda \ell_j^{(i)}\}$ with a generic collection $\{\ell_j^{(i)}\} \in \mathcal{L}_{\mathbf{k}}$.*

Corollary 3. *The set of collections of holomorphic 1-forms with a non-isolated special point at the origin has infinite codimension in the space of all holomorphic collections.*

Let $\{\omega_j^{(i)}\}$ be a collection of germs of 1-forms on $(V, 0)$ with an isolated special point at the origin. Let $\nu : \hat{V} \rightarrow V$ be the Nash transformation of the variety $V \subset B_\varepsilon(0)$ (see Section 5). The collection of 1-forms $\{\omega_j^{(i)}\}$ gives rise to a section $\hat{\omega}$ of the bundle

$$\hat{\mathbb{T}} = \bigoplus_{i=1}^s \bigoplus_{j=1}^{n-k_i+1} \hat{T}_{i,j}^*$$

where $\hat{T}_{i,j}^*$ are copies of the dual Nash bundle \hat{T}^* over the Nash transform \hat{V} numbered by indices i and j . Let $\hat{\mathbb{D}} \subset \hat{\mathbb{T}}$ be the set of pairs $(x, \{\alpha_j^{(i)}\})$ where $x \in \hat{V}$ and the collection $\{\alpha_j^{(i)}\}$ of elements of \hat{T}_x^* (i.e. of linear functions on \hat{T}_x) is such that $\alpha_1^{(i)}, \dots, \alpha_{n-k_i+1}^{(i)}$ are linearly dependent for each $i = 1, \dots, s$. The image of the section $\hat{\omega}$ does not intersect $\hat{\mathbb{D}}$ outside of the preimage $\nu^{-1}(0) \subset \hat{V}$ of the origin. The map $\hat{\mathbb{T}} \setminus \hat{\mathbb{D}} \rightarrow \hat{V}$ is a fibre bundle. The fibre $W_x = \hat{\mathbb{T}} \setminus \hat{\mathbb{D}}$ of it is $(2n-2)$ -connected, its homology group $H_{2n-1}(W_x; \mathbb{Z})$ is isomorphic to \mathbb{Z} and has a natural generator (see above). The latter fact implies that the fibre bundle $\hat{\mathbb{T}} \setminus \hat{\mathbb{D}} \rightarrow \hat{V}$ is homotopically simple in dimension $2n-1$, i.e. the fundamental group $\pi_1(\hat{V})$ of the base acts trivially on the homotopy group $\pi_{2n-1}(W_x)$ of the fibre, the last one being isomorphic to the homology group $H_{2n-1}(W_x)$: see, e.g., [St51].

Definition: The *local Chern obstruction* $\text{Ch}_{V,0}\{\omega_j^{(i)}\}$ of the collections of germs of 1-forms $\{\omega_j^{(i)}\}$ on $(V, 0)$ at the origin is the (primary) obstruction to extend the section $\hat{\omega}$ of the fibre bundle $\hat{\mathbb{T}} \setminus \hat{\mathbb{D}} \rightarrow \hat{V}$ from the preimage of a neighbourhood of the sphere $S_\varepsilon = \partial B_\varepsilon$ to \hat{V} , more precisely its value

(as an element of the homology group $H^{2n}(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon); \mathbb{Z})$ on the fundamental class of the pair $(\nu^{-1}(V \cap B_\varepsilon), \nu^{-1}(V \cap S_\varepsilon))$).

The definition of the local Chern obstruction $\text{Ch}_{V,0} \{\omega_j^{(i)}\}$ can be reformulated in the following way. Let $\mathcal{D}_V^{\mathbf{k}} \subset \mathbb{C}^N \times \mathcal{L}^{\mathbf{k}}$ be the closure of the set of pairs $(x, \{\ell_j^{(i)}\})$ such that $x \in V_{\text{reg}}$ and the restrictions of the linear functions $\ell_1^{(i)}, \dots, \ell_{n-k_i+1}^{(i)}$ to $T_x V_{\text{reg}} \subset \mathbb{C}^N$ are linearly dependent for each $i = 1, \dots, s$. (For $s = 1$, $\mathbf{k} = \{n\}$, $\mathcal{D}_V^{\mathbf{k}}$ is the (non-projectivized) conormal space of V [Tei82].) The collection $\{\omega_j^{(i)}\}$ of germs of 1-forms on $(\mathbb{C}^N, 0)$ defines a section $\tilde{\omega}$ of the trivial fibre bundle $\mathbb{C}^N \times \mathcal{L}^{\mathbf{k}} \rightarrow \mathbb{C}^N$. Then

$$\text{Ch}_{V,0} \{\omega_j^{(i)}\} = (\tilde{\omega}(\mathbb{C}^N) \circ \mathcal{D}_V^{\mathbf{k}})_0$$

where $(\cdot \circ \cdot)_0$ is the intersection number at the origin in $\mathbb{C}^N \times \mathcal{L}^{\mathbf{k}}$. This description can be considered as a generalization of an expression of the local Euler obstruction as a micro-local intersection number defined in [KS90], see also [Schü03, Sections 5.0.3 and 5.2.1] and [Schü04].

Remark. On a smooth manifold V the local Chern obstruction $\text{Ch}_{V,0} \{\omega_j^{(i)}\}$ coincides with the index $\text{ind}_{V,0} \{\omega_j^{(i)}\}$ of the collection $\{\omega_j^{(i)}\}$ defined above.

Remark. The local Euler obstruction is defined for vector fields as well as for 1-forms. One can see that vector fields are not well adapted to a definition of the local Chern obstruction. A more or less direct version of the definition above for vector fields demands to consider vector fields on a singular variety $V \subset \mathbb{C}^N$ to be sections $X = X(x)$ of $T\mathbb{C}_V^N$ such that $X(x) \in T_x V \subset T_x \mathbb{C}^N$ ($\dim T_x V$ is not constant). (Traditionally vector fields tangent to smooth strata of the variety V are considered.) There exist only continuous (non-trivial, i.e. with $s > 1$) collections of such vector fields "on V " with isolated special points, but not holomorphic ones.

Remark. The definition of the local Chern obstruction $\text{Ch}_{V,0} \{\omega_j^{(i)}\}$ may also be formulated in terms of a collection $\{\omega^{(i)}\}$ of germs of 1-forms with values in vector spaces L_i of dimensions $n - k_i + 1$. Therefore (via differentials) it is also defined for a collection $\{f^{(i)}\}$ of germs of maps $f^{(i)} : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^{n-k_i+1}, 0)$ (just as the Euler obstruction is defined for a germ of a function).

Being a (primary) obstruction, the local Chern obstruction satisfies the law of conservation of number, i.e. if a collection of 1-forms $\{\tilde{\omega}_j^{(i)}\}$ is a deformation of the collection $\{\omega_j^{(i)}\}$ and has isolated special points on V , then

$$\text{Ch}_{V,0} \{\omega_j^{(i)}\} = \sum \text{Ch}_{V,Q} \{\tilde{\omega}_j^{(i)}\}$$

where the sum on the right hand side is over all special points Q of the collection $\{\tilde{\omega}_j^{(i)}\}$ on V in a neighbourhood of the origin. With Corollary 2 this implies the following statements.

Proposition 8. *The local Chern obstruction $\text{Ch}_{V,0}\{\omega_j^{(i)}\}$ of a collection $\{\omega_j^{(i)}\}$ of germs of holomorphic 1-forms is equal to the number of special points on V of a generic (holomorphic) deformation of the collection.*

This statement is an analogue of Proposition 2.3 in [STV05b].

Proposition 9. *If a collection $\{\omega_j^{(i)}\}$ of 1-forms on a compact (say, projective) variety V has only isolated special points, then the sum of local Chern obstructions of the collection $\{\omega_j^{(i)}\}$ at these points does not depend on the collection and therefore is an invariant of the variety.*

It is reasonable to consider this sum as $((-1)^n$ times) the corresponding Chern number of the singular variety V .

Let $(V, 0)$ be an isolated complete intersection singularity. The fact that both the Chern obstruction and the index of a collection $\{\omega_j^{(i)}\}$ of 1-forms satisfy the law of conservation of number and they coincide on a smooth manifold yields the following statement.

Proposition 10. *For a collection $\{\omega_j^{(i)}\}$ on an isolated complete intersection singularity $(V, 0)$ the difference*

$$\text{ind}_{V,0}\{\omega_j^{(i)}\} - \text{Ch}_{V,0}\{\omega_j^{(i)}\}$$

does not depend on the collection and therefore is an invariant of the germ of the variety.

Since by Proposition 9 $\text{Ch}_{V,0}\{\ell_j^{(i)}\} = 0$ for a generic collection $\{\ell_j^{(i)}\}$ of linear functions on \mathbb{C}^N , one has the following statement.

Corollary 4. *One has*

$$\text{Ch}_{V,0}\{\omega_j^{(i)}\} = \text{ind}_{V,0}\{\omega_j^{(i)}\} - \text{ind}_{V,0}\{\ell_j^{(i)}\}$$

for a generic collection $\{\ell_j^{(i)}\}$ of linear functions on \mathbb{C}^N .

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